A MANUAL

OF

CIVIL ENGINEERING.

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BY

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With Numerous Diagrams.

TWENTY-THIRD EDITION, THOROUGHLY REVISED.

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PREFACE TO THE FIRST EDITION.

THIS work is divided into three parts. The first relates to those branches of the operations of engineering which depend on geometrical principles alone: that is to say, SURVEYING, LEVELLING, and the SETTING-OUT of works, comprehended under the general name of ENGINEERING GEODESY, or FIELD-WORK. The second part relates to the properties of the MATERIALS used in engineering works, such as earth, stone, timber, and iron, and the art of forming them into STRUCTURES of different kinds, such as excavations, embankments, bridges, &c. The third part, under the head of COMBINED STRUCTURES, sets forth the principles according to which the structures described in the second part are combined into extensive works of engineering, such as Roads, Railways, River Improvements, Water-Works, Canals, Sea Defences, Harbours, &c.

The first chapter of the second part, entitled a *Summary of the Principles of Stability and Strength*, forms not so much an integral part of the book, as a collection of mechanical principles and formulæ, introduced for the sake of being conveniently referred to in the subsequent chapters, so as to prevent their being encumbered with mathematical investigations to a greater extent than is absolutely necessary.

The third part, so far as the details of the designing and execution of works are concerned, consists, to a great extent, of references to the first and second parts, its special object being to explain those principles which are peculiar to each class of great works of engineering, and which regulate the general plan of such works.

The tables of the strength of materials at the end of the volume give, as regards iron and stone, average and extreme results only. Detailed information as to the strength of different kinds of stone and iron is given in the course of the text, under the proper headings.

I have, throughout the book, adhered to a systematic arrangement as far as was practicable, and have only departed from it in a few instances, when it became necessary to introduce questions that had arisen, or facts that had been ascertained, after the completion of the part of the work to which they properly belonged. In drawing up the table of contents and the alphabetical index care has been taken to show where such detached pieces of information are to be found.

W. J. M. R.

GLASGOW COLLEGE, *6th January*, 1862.

PREFACE TO THE TWENTY-THIRD EDITION.

THE Twenty-Third Edition has been carefully revised, and additional new matter has been added bearing on Civil Engineering Science and Practice.

W. J. M.

GLASGOW, *January*, 1907.

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PART I.

OF ENGINEERING GEODESY; OR, SURVEYING, LEVELLING, AND SETTING-OUT.

CHAPTER I.

GENERAL EXPLANATIONS.

1. **Surveying, Levelling, and Setting-out,** comprehend the principal operations of Engineering Geodesy: the object of surveying and levelling being to make a representation on paper of the ground on which the proposed engineering work is to be executed; and the object of setting-out being, to mark upon the ground the situation of the proposed work preparatory to its execution.

The term "surveying," when used in a comprehensive sense, includes levelling; but in a restricted sense, *surveying* is used to denote the art of ascertaining and representing the form of the ground and the relative positions of objects upon it, as projected on a horizontal surface; and *levelling*, to denote the art of ascertaining and representing the relative elevations of different parts of the ground, and of objects upon it.

2. **Plan and Section.**—The results of surveying, laid down on paper by the operations of "plotting" and drawing, constitute a *plan* or *ground plan;* those of levelling are usually laid down in the form of a *vertical section*, called more briefly a *section* (although there are other ways of representing them, as will afterwards be explained).

A plan is a miniature representation of the ground and the objects upon it, and of the proposed engineering work, as projected on a horizontal surface, that surface being represented by the surface of the paper on which the plan is drawn. A plan differs from a map chiefly in the scale on which it is drawn, the scale of a plan being large enough to serve for the designing of engineering works, while that of a map is so small as to make it serviceable for the purposes of travelling and geography only.

A vertical section shows the figure of a certain line or track on the natural surface of the ground, and of the proposed work to be executed along that line, and sometimes also that of the internal strata, as projected on a vertical surface,—that vertical surface being

represented by the surface of the paper on which the section is drawn. A certain straight line on that paper, called the "*datum-line*," represents a fixed horizontal surface at any convenient height above or depth below some fixed and known point, called the "*datum-point*." Lines parallel to the datum line represent in miniature, distances measured horizontally along the line or track on the earth's surface to which the section relates. Lines perpendicular to the datum-line represent in miniature, heights above or depths below the datum horizontal surface. The natural surface of the ground, and the proposed work, are represented by lines, straight, curved, or angular, which at each point are at the proper vertical distance from the datum line.

In the same section the scale for horizontal distances and the scale for heights may be different, if convenience requires it, as will afterwards be more fully explained.

3. **A Horizontal Surface** is a surface which is everywhere perpendicular to the direction of the force of gravity; such as the surface of a piece of still water. Its true figure is very nearly that of a spheroid. For a horizontal surface at the mean level of the sea, the dimensions of that spheroid are as follows, according to recent calculations:—*

	Feet.	Statute Miles.
Polar axis,..	41,707,536 =	7899 155
Mean equatorial diameter,..........	41,847,662 =	7925 694
Difference, or polar flattening,......	140,126 =	26 539

The portions of the earth's surface represented by plans for engineering purposes are usually so small compared with the whole earth, that a horizontal surface may, in most cases, be treated as if it were plane, without any error of practical importance. In plans, a flat piece of paper, and in vertical sections, a straight line, represent a horizontal surface with as much accuracy as is practicable. In many cases in which it is necessary to take the earth's curvature into account, the ellipticity or polar flattening may be neglected, and the figure of a horizontal surface may be treated as if it were a sphere of the same *mean diameter* with the spheroid before described; that is to say, very nearly

41,778,000 feet 13,926,000 yards = 7,912½ statute miles.

4. **Measures of Length.**—The standard measure of length established by law in Britain is the *yard*, being the distance, at the temperature of 62° of Fahrenheit's thermometer, and under the

* According to Captain Clarke, in the *Memoirs of the Royal Astronomical Society*, Vol. XXIX., the greatest and least equatorial axes are respectively 41,852,970 feet, and 41,842,354 feet; and the longitude of the greatest axis is about 14° E. of Greenwich. The polar axis, as Sir J. F. W. Herschel has pointed out, is almost exactly 500,500,000 inches

mean atmospheric pressure, between two marks on a certain bar which is kept in the office of the Exchequer, at Westminster.

In addition to the yard, the following units of length are employed for purposes of civil engineering in Britain:—

The **Inch**, one thirty-sixth part of the standard yard; with binary, decimal, or duodecimal subdivisions.

The **Foot**, one-third part of the standard yard; with decimal or duodecimal subdivisions.

The **Fathom** of two yards.

The **Chain** of 66 feet or 22 yards; divided into four *poles* of $5\frac{1}{2}$ yards, and 100 *links* of 7·92 inches.

The **Statute Mile** of 1,760 yards = 5,280 feet = 80 chains, divided into 8 *furlongs*. To these may be added, in cases of harbour engineering—

The **Nautical** or **Sea Mile**, being the length of one minute of a degree of latitude at the mean level of the sea. The length of this mile varies in different latitudes, from about 6,107 feet at the poles to about 6,045 feet at the equator, its mean value being nearly 6,076 feet, or 1·1508 statute mile. A value commonly taken for the nautical mile is that of a minute of longitude at the equator, or 6086 feet = 1·1527 statute mile. The nautical mile is sometimes subdivided into 10 *cables*, and 1,000 *fathoms;* the fathom thus obtained being, on an average, about $\frac{1}{80}$th longer than the common fathom.

Amongst obsolete measures of distance the following may be mentioned, as they occasionally occur in old plans:—

The Irish *Perch* of 7 yards, being greater than the imperial perch in the proportion of 14 to 11.

The Irish *Mile* of 320 Irish perches = 2,240 yards = 6,720 feet, bearing to the statute mile the same proportion of 14 to 11.

The Scottish *Ell* of 37·06 imperial inches.

The Scottish *Fall* of 6 ells, or 18·53 imperial feet.

The Scottish *Mile* of 1,920 ells = 5929·6 feet.

Each of these miles is divided, like the statute mile, into 8 furlongs, and 80 chains, so that the Irish, Scottish, and imperial mile, furlong, and chain, bear to each other the proportions—

$$6720 : 5929{\cdot}6 : 5280$$
$$:: 1{\cdot}27 : 1{\cdot}123 : 1{\cdot}000$$

The French measures of length are all decimal multiples and submultiples of the METRE, which is approximately one ten-millionth part of the distance from one of the earth's poles to the equator. The value of the mètre in British measures is

3·2808693 feet, or 39·37043 inches.

The **Kilometre** of 1,000 mètres, or 3280·8992 British feet, is 0·621383 of a statute mile.

For further information of the same kind, see the Comparative Table of French and British Measures at the end of the volume.

5. The **Measures of Area** used in British civil engineering are—

The **Square Inch.**

The **Square Foot** of 144 square inches.

The **Square Yard** of 9 square feet.

The **Acre** of 10 *square chains*, or 100,000 *square links*, or 4,840 *square yards*, subdivided either decimally, or into 4 *roods* of 1,210 square yards, and 160 *perches* of $30\frac{1}{4}$ square yards.

The **Square Mile** of 640 acres, or 3,097,600 square yards, or 27,878,400 square feet.

The Irish *acre*, subdivided into 4 *roods* and 160 *perches*, and the Scottish *acre*, subdivided into 4 *roods* and 160 *falls*, bear to the imperial acre proportions which are the squares of the proportions borne by the Irish and Scottish miles respectively to the statute mile; that is to say,

Irish acre : Imperial acre : : 196 : 121;
Also, Irish acre : Scottish acre : Imperial acre
: : 1·6198 : 1·2612 : 1·0000 nearly.

6. The **Measures of Volume** used in British civil engineering are—

The **Cubic Inch.**

The **Cubic Foot** of 1,728 cubic inches.

The **Cubic Yard** of 27 cubic feet.

In the engineering of water-works, the **Gallon** is used in stating quantities of water. Its statutory value is

277·274 cubic inches, or 0·16046 cubic foot;

but it is convenient in calculation, and in general sufficiently accurate for purposes of water supply, to use the approximate values,

One gallon ... = 0·16 cubic foot, nearly; and
One cubic foot = $6\frac{1}{4}$ gallons, nearly.

Other special measures of volume are employed for certain kinds of materials and work; but these will be explained further on.

7. **Scales for Plans.**—The scale on which a plan is drawn means the proportion which distances, as represented on the plan, bear to the corresponding distances on the ground. Amongst continental European nations it is customary to express that proportion by means of a fraction, such as 1-10,000th. In Britain, it is customary to refer to two units of length, a short unit for the paper, and a

long unit for the ground. For example—"*six inches to one mile*" expresses the scale which, according to the continental system, would be called 1-10,560th. Amongst continental nations, also, the scales most commonly used are those in which the proportion of the dimensions of the plan to those of the ground is some exact decimal fraction, such as 1-10,000th = ·0001, 1-2,500th = ·0004, 1-500th = ·002, &c.; in Britain, the scales most commonly used are those in which a distance of a certain number of miles, chains, or feet on the ground is represented by a distance of a certain number of inches, or aliquot parts of an inch, on the paper.

The magnitude of the scale which is best suited for the plan of a particular survey varies according to the minuteness and complexity of the objects to be represented. Thus, a larger scale is required in plans of towns than in those of the open country; and the smaller and more intricate the buildings and the divisions of property are, the larger should the scale be; and a plan to be used in the final designing and setting-out of works should be on a larger scale than one to be used for the selection of a line of communication, and for preliminary or parliamentary purposes. (See p. 803.)

The following table enumerates some of the scales for plans most commonly used in Britain, together with a statement of the purposes to which they are best adapted:—

Ordinary Designation of Scale.	Fraction of real Dimensions.	
(1.) 1 inch to a mile,.	$\frac{1}{63,360}$	Scale of the smaller ordnance maps of Britain. This scale is well adapted for maps to be used in exploring the country.
(2.) 4 inches to a mile,..	$\frac{1}{15,840}$	Smallest scale permitted by the standing orders of parliament for the deposited plans of proposed works.
(3.) 6 inches to a mile,..	$\frac{1}{10,560}$	Scale of the larger ordnance maps of Great Britain and Ireland. This scale, being just large enough to show buildings, roads, and other important objects distinctly in their true forms and proportions, and at the same time small enough to enable the eye of the engineer to embrace the plan of a considerable extent of country at one view, is on the whole the best adapted for the selection of lines for engineering works, and for parliamentary plans and preliminary estimates.
(4.) 6·336 inches to a mile,..	$\frac{1}{10,000}$	Decimal scale possessing the same advantages.

Ordinary Designation of Scale.	Fraction of real Dimensions.	Use.
(5.) 400 feet to an inch,.......	$\frac{1}{4,800}$	Smallest scale permitted by the standing orders of parliament for "enlarged plans" of buildings and of land within the curtilage of buildings.
(6.) 6 chains to an inch,......	$\frac{1}{4,752}$	Scale answering the same purpose.
(7.) 15·84 inches to a mile,....	$\frac{1}{4,000}$	Scales well suited for the working surveys and land plans of great engineering works, and for enlarged parliamentary plans.
(8.) 5 chains to an inch, or 16 inches to a mile,	$\frac{1}{3,960}$	(Scale 8 is that prescribed in the standing orders of parliament for "cross sections" of proposed railways, showing alterations of roads.)
(9.) 25 344 inches to a mile.	$\frac{1}{2,500}$	Scale of plans of part of the ordnance survey of Britain, from which the maps before mentioned are reduced. Well adapted for land plans of engineering works and plans of estates
(10.) 200 feet to an inch,......	$\frac{1}{2,400}$	Scale suited for similar purposes. Smallest scale prescribed by law for land or contract plans in Ireland.
(11.) 3 chains to an inch,......	$\frac{1}{2,376}$	Scale of the Tithe Commissioners' plans. Suited for the same purposes as the above.
(12.) 100 feet to an inch, ...	$\frac{1}{1,200}$	Scale suited for plans of towns, when not very intricate
(13.) 88 feet to an inch, or 60 inches to a mile,	$\frac{1}{1,056}$	Scale of ordnance plans of the less intricately built towns.
(14.) 63·36 inches to a mile,..	$\frac{1}{1,000}$	Decimal scale having the same properties.
(15.) 44 feet to an inch, or 120 inches to a mile,	$\frac{1}{528}$	Scale of ordnance plans of the more intricately built towns.
(16.) 126 72 inches to a mile,	$\frac{1}{500}$	Decimal scale having the same properties.
(17.) 30 feet to an inch,........	$\frac{1}{360}$	Scales for special purposes.
(18.) 20 feet to an inch,........	$\frac{1}{240}$	
(19.) 10 feet to an inch,........	$\frac{1}{120}$	
&c.	&c.	

8. **Scales for Sections.**—Except in a few cases of rare occurrence, the *scale for horizontal distances* on a section should be the same with the scale of the plan with which it corresponds. One of the exceptions is that of the parliamentary section of a road upon the level or position of which it is intended to make an alteration for the purpose of carrying a railway across it, whether over or under; in this case, the horizontal scale of the section, as prescribed by the standing orders, is to be *five chains to an inch* (see No. 8 in the table of the last article). The plan *may* be on the same scale, but not necessarily so; in fact, its scale in general is much smaller.

The *vertical scale*, or *scale for heights*, is almost always much greater than the horizontal scale, because the differences of elevation between points on the ground are in general much smaller than their distances apart, and require to be represented on a greater scale on paper, in order that they may be equally conspicuous to the eye; and also, because in the execution of engineering works, accuracy in levels is of more importance than accuracy in horizontal position, and vertical heights should be represented with greater precision than horizontal distances. The proportion in which the vertical scale is greater than the horizontal scale is called the *exaggeration* of the scale. The following table gives some examples:

Ordinary Designation of Vertical Scale.	Fraction of real Height.	Horizontal Scales with which the Vertical Scale is usually combined.	Exaggeration.	Use.
(1.) 100 feet to an inch,	$\frac{1}{1,200}$	$\frac{1}{15,840}$ to $\frac{1}{10,560}$	From 13·2 to 8·8	Smallest scale permitted by the standing orders of parliament for sections of proposed works.
(2.) 40 feet to an inch,	$\frac{1}{480}$	$\frac{1}{4,800}$ to $\frac{1}{3,960}$	10 to 8·25	Smallest scale permitted by the standing orders of parliament for cross sections, showing alterations of roads.
(3.) 30 feet to an inch,	$\frac{1}{360}$	$\frac{1}{3,960}$ to $\frac{1}{2,376}$	11 to 6·6	Scales suitable for working sections.
(4.) 20 feet to an inch,	$\frac{1}{240}$	$\frac{1}{3,960}$ to $\frac{1}{2,376}$	16·5 to 9·9	
&c.	&c.	&c.	&c.	

Vertical sections *without exaggeration*, showing the horizontal and vertical dimensions of the ground in their real proportions to each other, are required at the sites of proposed large works in masonry, timber, and iron, such as viaducts. These sections are in general drawn on a larger scale than the vertical scale of the ordinary working sections.

9. **Methods in Surveying.**—There are two principal methods followed in surveying, each characterized by the elementary mathematical process which it involves: *the method of distances and offsets*, used for filling up the details of a survey, and the *method of triangles*, used chiefly for ascertaining the positions of certain *stations*, but occasionally applied to filling up the details also.

First Method—By Distances and Offsets.

In fig. 1, A is the representation on paper of a station, or fixed and marked point on the ground, and A D that of a line extending from A in a known direction. To ascertain and lay down the position of a point C relatively to A, a perpendicular is let fall on the ground from C upon A D, meeting that line in B; the *distance* A B and *offset* B C are measured, and these being laid down on the plan to a suitable scale, the point C on the plan which represents C on the ground is marked or *plotted*. In some cases the angle at B may be some measured oblique angle instead of a right angle; but in most cases it is a right angle. This is the method of surveying by distances and offsets, and is that by which the details of a survey are in almost all cases filled in.



Fig. 1.

The same figure may be taken as representing the elementary operation of levelling, if A D be held as marking the datum horizontal surface, and C B the height above that surface of a point C, whose horizontal distance from A, the commencement of the section, is A B.

Second Method—By Triangles.

A and B, upon the paper, represent two stations or points on the ground, whose relative position—that is, their distance apart, and the direction of the line joining them—has been ascertained. It is required to ascertain and lay down on the paper the position of a third point C relatively to those two. This is to be done by measuring any two out of the following four quantities:—

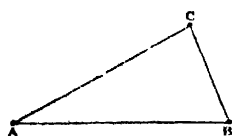


Fig. 2.

the distances A C and B C;—
the angles C A B and C B A,—

and plotting or laying down on the paper the representation either of the quantities actually measured, or of others calculated from them. The object of such calculation is in most cases to lay down the distances A C and B C on paper, when the angles at A and B have been measured on the ground; for on the ground, angles are more easily measured with precision than distances; and on paper, distances can be laid down more accurately than angles.

10. **Use of Trigonometry.**—The figure to be measured on the ground and laid down on the paper being in most cases a *triangle*, the branch of mathematics by which the necessary calculations are to be performed is that which relates to the figures and dimensions of triangles: that is, TRIGONOMETRY.

When the triangle formed by the three points is of such extent that the curvature of the earth may be neglected, its sides are sensibly straight lines, and the rules of *Plane Trigonometry* are to be used. When the curvature of the earth has a sensible effect, the sides of the triangle are to be considered as being nearly arcs of circles, of a radius equal to that of the earth, and recourse must be had to *Spherical Trigonometry*. This, however, is of rare occurrence in surveys made expressly for engineering purposes. The principles of spherical trigonometry are also occasionally required, when an angle has been measured on an inclined plane, to compute the corresponding angle as projected on a horizontal plane.

In Chapter III. will be given a summary of those trigonometrical formulæ which are useful in surveying.

11. The **General Order of Operations in Engineering Geodesy** is the following, or nearly so:—

I. The *reconnaissance* or *exploring* of the country by the engineer, with a view to ascertaining in a general way the facilities which it affords for the proposed work, and determining approximately the best site or course for that work. In this process the engineer will pay attention to the geological structure of the ground, and the sources from which useful materials may be obtained: he will be aided by obtaining the best existing maps or plans upon a suitable scale, if any such are to be had, and by the taking of—

II. *Flying levels.*—These are observations for ascertaining the elevations of detached points of primary importance as regards the practicability and cost of the work, and the selection of the line for it; such as passes across ridges and valleys, and points where structures of magnitude may be required.

The engineer having thus determined generally where his proposed work will be situated, proceeds to make a more definite selection of its site, by the aid of—

III. *Preliminary Trial Sections*, made by taking continuous lines of levels in which distances as well as heights are measured. These may, or may not, be accompanied by a *rough survey and plan*,—the necessity for which will depend very much on the character of the existing maps. The engineer is now enabled to determine the site of the work with a degree of precision depending on the care and skill that have been bestowed on the preliminary operations, and to fix accordingly what extent of ground is to be embraced in the—

IV. *Detailed Survey and Plan*, as to the conduct of which further remarks will be made in Article 12. The time, labour, and money expended on this survey will be the less, the greater the precision with which the best line has been found by means of the preliminary operations.

V. *Additional Trial Sections*, both longitudinal and transverse, are now to be made with the aid of the detailed plan, so as to fix exactly the best line for the proposed work that can be found.

VI. *Marking the Line.*—The line so fixed is to be drawn on the plan, and marked on the ground by stakes, or other suitable objects. (See Article 13.)

VII. The *Detailed Section* is now to be prepared by careful and accurate levellings, so as to exhibit a datum horizontal line, a line representing the surface of the ground, and a line, or lines, marking the levels of the proposed work. Certain heights and other information should be marked in figures, as will afterwards be explained. (See Articles 14, 15, and 16.)

VIII. *Trial-Pits and Borings* will be proceeded with, while the levelling for the detailed section is in progress, in order to ascertain the strata of the ground. Borings are the less costly, in time, labour, and damage to the ground; but pits are the more satisfactory to the engineer and the contractor. The results of the trial-pits and borings may be marked on a plan and section for the use of the engineer. (See Article 17.) Further remarks will be made on these matters under the head of earthwork.

IX. *Designs and Estimates.*—The engineer will now design the structures required for the proposed work with sufficient precision to enable him to estimate their probable cost. (See Article 17.)

X. *Parliamentary Proceedings.*—In the event of its being necessary to apply for an act of parliament for the execution of the work, a plan and section and book of reference to the plan will be prepared, and copies of them deposited in certain public offices, in conformity with the standing orders of the House of Lords, and also with those of the House of Commons. No attempt is made in this treatise to give any summary of those standing orders, because, as they are liable to be amended and

added to in each session of parliament, the only means of ensuring compliance with them is for the engineer to provide himself with a copy of the standing orders for the session during which the act is to be applied for. Those for a previous session, even for that immediately preceding, are unsafe guides.

XI. *Improving Lines and Levels, under Powers of Deviation.*—In the first preparation of the plan and section of a work requiring the authority of parliament, there is seldom or never time to select the best line and levels with precision. In order to afford an opportunity for afterwards amending the line and levels, powers of deviating from those shown on the parliamentary plan and section are taken, the extent of the power of lateral deviation being indicated on the plan by dotted lines. The usual extent of those powers of deviation is, laterally, 100 yards either way in the country, and 10 yards either way in towns; and vertically, five feet upwards or downwards in the country, and two feet upwards or downwards in towns; but greater or less powers are conferred in special cases. After the act of parliament has been obtained, the engineer will avail himself of the power of deviation to make the work more economical, or otherwise to improve it.

The following four operations will then proceed together:—

XII. *Survey for Land Plans.*—If, as is often the case, the previous survey referred to under Operation IV., has been executed too hastily, or plotted on too small a scale, to serve for the plans that are to be used in the purchase of land and execution of the work, a more accurate survey must now be made for that purpose; but this new survey being confined to the ground finally selected for the site of the work, will be of comparatively small extent. (See Article 18.)

XIII. *Ranging and Setting-out the Line,* consists in marking, by stakes or otherwise on the ground, the centre line of the proposed work, as finally fixed.

XIV. *Working Sections* are prepared by taking, with great care and precision, the levels of the ground along the finally selected centre line, and as many lines of transverse sections as may be necessary, plotting the results on a sufficiently large scale (see Article 8, p. 7), and drawing on the sections of the ground so made, lines to represent the intended levels of the work. (See Articles 14, 15, and 16.)

XV. *Setting-out the Breadths of Land* required for the work is performed both on the ground and on the land plans after those breadths have been calculated.

The land required can now be fenced, and the execution of the work proceeded with.

12. Order of Operations in the Detailed Survey.—It will now be

stated, in greater detail, what steps are taken in making the survey referred to under Head IV. of Article 11, p. 10.

(*a.*) *Selecting Principal Stations.*—The surveyor, making a general exploration of the ground to be surveyed, will choose a series of stations placed generally on the highest and most open ground; so that each station may command as extensive a view as possible of the ground to be surveyed, and that a pole or other signal placed at each station may be distinctly visible from the neighbouring stations. These stations should also be chosen so that the imaginary lines connecting them with each other, and with a series of conspicuous objects in their neighbourhood, such as towers and spires, may cover the district to be surveyed with a network of large triangles, having no angle less than 30°, or more than 150°; two angles at least of each triangle being accessible stations.

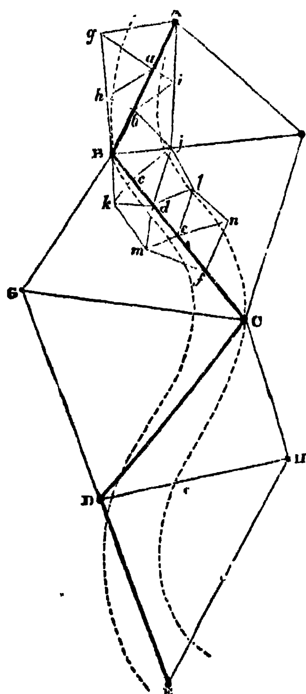


Fig. 3.

With the exception of harbours, most great engineering works are long lines of communication, such as railways, roads, and canals; and the survey required for a work of that sort embraces, in general, a long narrow band of country, usually about a quarter of a mile, and seldom more than half-a-mile wide. Let the two dotted lines in fig. 3 represent part of the band of country to be surveyed; the principal stations, A, B, C, D, E, &c., are to be chosen so as to form the junctions of a series of straight lines running along that band, each line as long as may be practicable consistently with obtaining good points for stations. These are called *base lines*, or *principal station lines.* The network of triangles is to be completed by selecting a series of lateral objects, F, G, H, &c., which may be high buildings, conspicuous trees, &c.

The principal stations are to be marked permanently by stakes, and temporarily, when required, by poles and flags.

(*b.*) *Ranging Principal Station Lines.*—When the lines are of great length, or have uneven ground and other obstacles in their

course, it may be necessary to mark intermediate points in them by stakes and poles, as well as the extremities. This is always necessary when there are parts of a station line from which its ends are not visible.

(*c.*) *Main Triangulation. Chaining Base Lines.*—The survey of the network of great triangles might be made by measuring one base line only, and finding the lengths of all the other sides of triangles by calculation from their angles. But for the purposes of the long narrow surveys required for engineering projects, it is more convenient to measure each of the principal station lines AB, BC, CD, &c., by the chain, in order to ascertain the positions of intermediate points suitable for secondary stations, and also of the points where the principal station-lines cross roads, fences, streams, and other objects on the ground. The term "base line" is specially applied to station lines which are thus directly measured. The relative directions of the base lines are determined by measuring the angles $\angle$ A B C, $\angle$ B C D, $\angle$ C D E, &c. The measurements of the angles made by distant objects with the base lines, such as $\angle$ A B F, $\angle$ F B C, $\angle$ C B G, $\angle$ B C F, $\angle$ B C G, $\angle$ G C D, $\angle$ D C H, &c., serve, by the aid of trigonometrical calculation, to check the accuracy of the other linear and angular measurements, as will afterwards be shown.

This combination of linear and angular measurement is called *traversing*. It has now been described as practised on a great scale, with principal station lines of several miles in length; but it is also practised on a small scale in surveying objects which are long, narrow, and winding, such as roads and streams.

(*d.*) *Secondary Triangles.*—The surveyor will choose a set of secondary stations, some in the course of the principal station lines; as, *a*, *b*, *c*, *d*, *e*, *f*,—others at convenient lateral points; as, *g*, *h*, *i*, *j*, *k*, *l*, *m*, *n*; the whole so situated that the lines connecting them, which form a network of smaller or *secondary* triangles, may lie sufficiently near to the fences, streams, buildings, and other objects of detail, to enable these to be surveyed from them by the first method of Article 9, p. 8,—that is, by distances and offsets.

(*e.*) *Survey of Details.*—This may be performed wholly by means of distances and offsets; but time and trouble may often be saved by the occasional use of angular measurements.

In order to save time, trouble, and money as far as possible, the five operations which have just been enumerated should be carried on either together or alternately.

13. **Distances, Levels, and other information written on Plan.**—When the plan shows the centre line of a railway, canal, or other line of communication, a scale of distances is to be marked along

the whole of its length, commencing at one of its ends or "termini." According to standing orders which have been in force for many years, that scale of distances on the plan of a proposed railway, is to show each *mile* and *furlong* from the commencement of the centre-line; all radii of curves which *do not exceed one mile* are to be written on the plan in *furlongs* and *chains;* and the lengths of proposed tunnels in *yards*. The information thus written on the plan is useful to the engineer, independently of its being prescribed. It is also useful to the engineer, although not prescribed, to have the levels of important points written on the plan, or shown by the aid of *contour lines* (which will be further explained afterwards), especially when the plan is to be used in selecting a line.

14. **Distances, Datum-point, Heights, and other information written on the Section.**—The horizontal datum-line of the section should have marked on it a scale of distances corresponding with those marked along the centre-line on the plan, in order that corresponding points on the plan and section may be readily found; and great care should be taken that horizontal distances on the plan and section exactly agree.

Alongside the datum-line on the section there should be a written statement of the elevation or depression of the horizontal surface which it represents as compared with what is called the "Datum fixed point," that is, a well-marked and easily found point on some permanent object, which (as prescribed in the standing orders of parliament) should be "near one of the termini" of the proposed work. The chief requisites of an object for that purpose are, permanence of position and easy identification; so that, on the whole, some portion of the masonry of a building (a public building, if possible), such as the upper side of a window-sill, plinth, or string-course, may be considered as the best. Door-sills are deficient in permanence because of their liability to be worn down by the feet of persons passing in and out; nevertheless, they are frequently used as datum-points, and not objected to. The upper surface of the rails of a railway at some specified point is often referred to as a datum, and considered sufficient, although its elevation is far from being permanent. Amongst objects utterly unsuitable for this purpose may be mentioned, all surfaces whose levels are continually changing, how slight soever the change may be, such as the "top water level" of a canal, and all *ideal* horizontal surfaces.

Amongst other information to be marked in writing on a section are, the heights of the principal parts of the proposed work above the horizontal datum-line, and in particular, in the case of a railway, those of the upper surface of the rails at the points where the

inclination varies; the several rates of inclination of proposed railways, and of roads to be altered for the purpose of making them; the greatest depths of cuttings and heights of embankments; the lengths of tunnels and viaducts; the alterations of level and inclination to be made in existing lines of communication; the character of the structures to be used for passing them, whether bridges over or under, or level crossings; and in the case of proposed bridges for existing roads, the width of roadway which they will provide, and if they pass over the roads, the height of headroom. So far as those items of information are required by the standing orders of parliament, reference must be made for details to those standing orders themselves, as has been already stated under Head X. of Article 11.

A working section should state, in writing, the level of the ground, the level of the proposed work, and the height of embankment or depth of cutting, at every point of the ground whose level has been taken; those quantities being found by calculation, not by measurement on the paper. It should also state the positions and levels of all "Bench marks."

15. **Bench Marks** are fixed objects whose levels are known,—in fact, subordinate datum-points,—distributed along the course of the intended work, at distances of from half-a-mile to a mile, and near the sites of all intended structures of importance, such as bridges. If suitable existing objects cannot be found, the heads of large stakes driven for the purpose will answer. They should be placed where they will not be disturbed during the execution of the work.

16. **Checking Levels** consists in taking the levels of points over again, to test the correctness of previous levelling. In preliminary and parliamentary sections, the levels of the more important points only, such as summits of hills and bottoms of valleys, crossings of existing lines of communication, and bench marks, require to be checked; for working sections, every level taken should be checked.

17. **Estimates and Borings marked on Plan and Section.**—It is useful to the engineer to have a copy of the plan and section of a proposed work on which the results of trial-pits and borings are marked, and the estimated cost of each part of the work written opposite to its position on the paper.

18. **Centre Line as a Base for Land-Plan Survey.**—When the centre line of a proposed railway has been carefully ranged and staked out, it may be used, whether straight or curved, as a base for the secondary triangulation of the survey for the land-plans, the great triangulation being dispensed with, and each stake regarded as a station in the survey.

19. **Damage to Property to be avoided.**—All operations of engi-

neering field-work ought to be so conducted as to do as little damage as possible to the property traversed.

20. **Arrangement of the ensuing Chapters.**—The operations of surveying, levelling, and setting-out, having been enumerated and explained in a general way in the present chapter, the remaining chapters of this part will be devoted to the explanation of details relative to certain branches of the subject, in the following order:—

Surveying with the chain.
Surveying by angular measurements.
Levelling.
Setting-out works.
Marine surveying.
Copying, enlarging, and reducing plans.

The explanation of some of the peculiarities of surveys for particular classes of works will be reserved until those works themselves come to be considered.

CHAPTER II.

OF SURVEYING WITH THE CHAIN.

21. **Marks and Signals.**—The marks fixed at stations to enable them to be readily found are usually stakes, of size and strength sufficient to guard against the risk of their being disturbed. In most cases they should be driven to the head, or nearly so. If, for a particular station-mark, greater permanence is desired than can be obtained by means of a stake, a block of stone may be used, having a cross cut on its upper surface.

When a mark fixed at the station itself would be liable to be disturbed, four stakes may be driven so that the intersection of the straight lines joining them diagonally may mark the station; or two or more stakes may be driven, and the distances of the station from them measured and noted down; or the distance of the station from any two or more well-defined permanent objects, such as corners of buildings, may be measured and noted down; or if two permanent objects can be found which lie in one straight line with the station, that fact can be noted, together with the distance of the station from one of the objects. The points where station-lines cross fences are marked by notches upon timber and grooves upon stone.

The signals set up at stations to make them visible from a distance usually consist of poles, with or without flags. Ordinary poles, to be carried about in the field, may be from six to nine feet long, painted in alternate lengths of black and white, and shod with iron. For flags, although white is the colour that is seen farthest, red is more generally employed, as being more easily distinguished from surrounding objects by those who have no defect in the perception of colour. To mark the ends of long station-lines, poles of greater lengths, such as twenty or thirty feet, are often required; these generally need rope stays to keep them upright.

Great care should be taken to set up and keep all poles in a truly vertical position; and tall permanent poles should be adjusted by means of a plumb-line.

For the temporary marking of points in surveying details, bits of paper are used, held in cleft sticks. These are called "whites."

To facilitate the ranging of long station-lines, it is useful to choose them, when opportunities occur, so as to run directly towards some conspicuous existing object, such as a tree, a spire, or a large chimney.

22. **The Surveying Chain.**—For measuring with extraordinary accuracy the bases of national trigonometrical surveys, rods of glass and of metal have been used—a correction for expansion by heat being made either by calculation or by mechanism: also, steel chains, made of flat links connected at the ends by pins, and supported in accurately levelled troughs, the tension being maintained constant by a weight hanging over a pulley, and the correction for expansion made by calculation.

In ordinary surveys for engineering works so great a degree of accuracy is unnecessary; and the instrument generally used for measuring distances is the common surveying chain, which consists of one hundred straight links of iron or steel wire of equal length, having eyes on their ends, and connected together by oval rings. There are usually three of those rings between each pair of straight links. The joints of the rings, and those of the eyes of the links, should be welded: the chain is thus rendered much less liable to stretch than if those joints are open. Each distance of ten links from either end of the chain is marked by a peculiarly shaped piece of brass, so that the mark at ninety links from one end is similar to that at ten links, that at eighty links to that at twenty, and so on, the middle of the chain being marked by a round piece of brass. At each end of the chain is a handle.

The chain should measure its correct length from *outside to outside of the handles.*

As every chain which is in daily use in the field is liable to have its length increased by the continual strain upon it, and diminished by the bending of the links, and by dirt getting into the rings, it ought to have its length tested every day by comparison with a "standard chain," used for the sole purpose of testing other chains, or with two marks on a wall, or on a pair of stakes, whose distance apart has been very accurately adjusted. The length of the working chain, when found to be erroneous, can be corrected by straightening the links and cleansing the rings, and by hammering the latter so as to make them longer or shorter as may be required.

The chains most commonly used in Britain are, "Gunter's Chain" of 66 feet (in which each link is ·66 of a foot or 7·92 inches), and the chain of 100 feet. The advantages of Gunter's chain are, its being an exact decimal fraction of a mile (one-eightieth, or ·0125), and the square described upon it being one-tenth of an acre. The 100-foot chain has the advantage of giving at once dimensions in feet, which are convenient in the calculation of quantities of work.

When a "chain" is spoken of without qualification, Gunter's chain is meant.

The chain is usually accompanied by ten skewers called "arrows," made of iron or steel wire, having a point at one end and a large ring at the other, marked with a piece of red cloth to make it visible from a distance. Some surveyors prefer to use, in chaining long lines, nineteen arrows, nine of iron or steel, and ten of brass.

The chain is carried by two men, called respectively the "leader" and the "follower." In measuring the length of a station-line, the follower, in a crouching attitude, holds one end at the commencement of the line, and the leader, carrying with him all the arrows, fixes his eyes on the object which marks the distant end of the line, and walks straight towards it, dragging the other end of the chain along with him. When the chain is tightened, the leader crouches down at one side of the line, holding near the ground an arrow exactly upright, in the same hand which grasps the handle of the chain. The follower sees that the chain is tight, straight, and unentangled, and directs the leader by words or gestures so as to make him stick the arrow into the ground exactly in the alignment.* The leader and follower then rise, and advance until the follower reaches the arrow that marks the end of the first chain-length, and proceed to lay off a second chain length and fix a second arrow as before, and so on. The follower picks up the arrows as he advances, so that by counting the arrows in his hand he can tell at any moment how many entire chain-lengths have been measured. On fixing the tenth arrow, the leader cries in a loud voice "ten," or "change;" the surveyor notes in his field-book that ten chains have been measured; the leader stands still until the follower has advanced to him and handed him the nine previously picked-up arrows; the follower holds his end of the chain at the mark made by the tenth arrow, which the leader (if there are ten arrows only) then picks up, and advances with all the ten arrows in his hand to commence the measurement of the next ten chains. If there are nine iron arrows and ten brass ones, the leader, having expended all the iron arrows in marking the first nine chains, marks the end of the tenth chain with a brass arrow; and when the follower comes up to him, takes only the nine iron arrows, leaving the brass arrow to be picked up by the follower when the next chain-length has been measured. In this case the follower, at any moment, can tell the number of entire tens of chains which have

* Mr. Haskoll (*Engineering Field-work*) judiciously recommends that words alone be used for this purpose, in order that the leader may fix his eyes on the arrow, and keep it exactly vertical, the follower directing him to move it to one side or the other by saying "to you" and "from you," and to fix it in the ground by the word "mark."

been chained by counting the brass arrows in his hand, and the number of chains over and above the entire tens of chains by counting the iron arrows; and thus a check is kept upon the number of entire tens of chains noted in the surveyor's field-book. At the end of each hundred chains the leader receives back all the brass arrows as well as the iron ones.

If the leader takes care while advancing to keep his eyes fixed on the signal at the distant end of the line, he will be able to drag the chain forward in the true alignment with very little direction from the follower.

The follower while advancing should allow the chain to slacken, and should take care to keep it clear of the arrow, and of objects which may entangle it.

As the chaining goes on, the surveyor notes the distances from the commencement at which the station-line crosses all fences, boundaries, banks of streams, sides of roads, and other objects to be shown on the plan; also where it crosses other station lines, and where points occur suitable for intermediate stations in the survey.

23. **Chaining on a Declivity—Reduction to the Level.**—In chaining up or down a slope, the distance actually measured must be reduced on the plan to the projection of that distance on a horizontal plane. The most convenient way of effecting this is by means of a *correction* in links and fractions of a link to be *deducted* from each chain. This correction being known, may be applied mechanically during the chaining, by pulling the chain forward at each chain-length through a distance equal to the required correction.

The following are various formulæ for computing the correction:—

When the angle of inclination has been measured by a "clinometer" or other angular instrument;

Correction in links per chain, = 100 × versed sine of inclination, (1.)

When the vertical fall in links for each chain of distance on the slope is known;

$$\text{Correction in links per chain} = 100 - \sqrt{10{,}000 - \text{fall}^2}; \quad (2.)$$

and when the slope is gentle, the following approximate formula will answer:—

$$\text{Correction in links per chain} = \frac{\text{fall}^2}{200} \text{ nearly.......}(3.)$$

To save calculation, most clinometers and theodolites have the correction for declivity marked on the "limb" or graduated arc on which angles in a vertical plane are measured.

Experienced surveyors learn to estimate this correction with considerable accuracy by the eye.

Its use may often be dispensed with by stretching the chain in a

horizontal position; the up-hill end touching the ground, and the point on the ground exactly below the down-hill end being found by means of a plumb-line, or a ranging pole held vertically; or by dropping an arrow or a stone. This process is called *stepping*, and may be carried on by half-chains or shorter distances, instead of whole chains, on very steep ground.

24. **Offsets** (to which reference has already been made in Article 9, Division I., p. 8) are ordinates or transverse distances, measured from known points in a station-line to objects whose position is to be ascertained; such as bends and intersections of fences, of the sides of roads, of the banks of streams, and of other boundaries, corners of buildings, and so forth. The surveyor notes in his field-book the distance in links from the commencement of the station-line at which the offset is made (A B, fig. 1, p. 8), and the length of the offset (B C in the same figure); the side of the page on which the latter is noted showing at which side of the station-line the offset lies, as will be further explained in Article 28.

Offsets are almost always at right-angles to the station-line. To ensure accuracy they should seldom exceed about one chain in length (although offsets of two or three chains may be made to boundaries which are nearly parallel to the station-line); and the secondary station-lines from which the details of the ground are surveyed should be laid out accordingly. The position and direction of short offsets may be laid off by the eye; but the longer offsets, especially if they run to important objects, should be laid off by letting fall a perpendicular from the object (at which, if necessary, a pole or a "white" may be placed) upon the station-line, by means of the "cross-staff" or of the "optical square."

The *Cross-Staff* is simply a staff with a spike on the lower end, and two pair of sights at right angles to each other at the upper end.

The *Optical Square*, which has almost superseded the cross-staff, is a brass box, containing two small silvered plate-glass mirrors, whose planes make with each other an angle of 45°; so that every ray of light which falls upon the first mirror, and is thence reflected to the second mirror, is again reflected from the second mirror in a direction at right angles to its original direction. A portion of the second mirror is unsilvered, so that the surveyor can see through it. He places himself on the station-line, and looks through the unsilvered glass towards the signal at one end of it, and then moves backwards and forwards along the station-line until he sees the reflected image of the lateral object apparently coinciding in direction with the signal on the station-line; the directions of those two objects are then at right angles, and the point on the ground directly below the optical square is the commencement of the offset required.

To adjust the optical square, make a rest for it by driving a picket or small post (which may be called A) four and a-half or five feet high, with a flat top. Set up a pole two or three chains off in any convenient direction (which pole may be called B); look towards it through the unsilvered glass; send an assistant to set up a second pole (C) in such a direction that its reflected image apparently coincides in direction with B. Then the lines A B and A C are or ought to be at right angles. In the same way, let the assistant set up a third pole, D, at the same angular distance from C, and a fourth pole, E, at the same angular distance from D. Then on looking directly towards E, if the optical square is correctly adjusted, the reflected image of B will be seen apparently coinciding in direction with E. Should it not be so, correct one quarter of the error by means of the adjusting screw which acts upon one of the mirrors, and repeat the whole operation until the adjustment is exact.

The purpose of an optical square may be answered by a *box-sextant*, the index being set to 90°. This instrument will be described in Chapter III. Lines at right angles to each other may sometimes be marked on the ground by setting out with the tape-line or chain a right angled triangle of any convenient dimensions, the proportions of the sides being determined by the principle, that the sum of the squares of the sides which enclose the right angle is equal to the square of the hypothenuse, or side opposite the right angle.

Amongst the proportions of whole numbers which fulfil that condition are the following:—

Sides enclosing the right angle		Hypothenuse.
3	4	5
5	12	13
7	24	25
8	15	17
20	21	29

The most useful of these proportions is the first and simplest,*

* The following is a general method for finding any number of sets of whole numbers which are proportional to the sides of right-angled triangles.

Choose any two numbers whatsoever, m and n, m being the greater; and if they are either both even, or both odd, make

$$x = mn; \quad y = \frac{m^2 - n^2}{2}; \quad z = \frac{m^2 + n^2}{2};$$

but if one is even and the other odd, multiply each of the above expressions by 2. Then,

$$x^2 + y^2 = z^2;$$

and x, y, and z are proportional to the three sides of a right-angled triangle, z corresponding to the hypothenuse.

When two persons are available to measure the lengths of offsets, either a second chain or a *Tape-line* may be used. The surveyor may measure offsets without assistance with the *offset-staff*,—a light and strong wooden pole tipped with brass or iron, ten links long from end to end, and divided into links.

25. **Oblique Offsets** may be made, if convenient, with the aid of an angular instrument, such as a box-sextant or a light theodolite, to measure the angles which they make with the station-line. But in surveying by linear measurements alone, oblique offsets are made in pairs from different points in the station-line to the same object, in order to determine its position with more accuracy than is attainable by a single rectangular offset. For example (see fig. 4.), the position of the object D is found by measuring to it a pair of offsets, B D, C D, from two different points, B and C, in the station-line A B C. This process, in fact, belongs to the method of surveying *by triangles*, B D C being a triangle of which the three sides are measured.

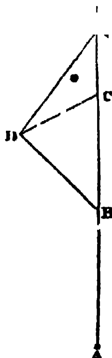


Fig. 4.

The nearer the angle between the two offsets, ∠ B D C, approaches to a right angle, the more accurately is the position of the object determined, and care should therefore be taken to make that angle neither very acute nor very obtuse.

If a check on the accuracy of the operation is desired, a third offset, E D, may be measured to the object from a third point, E, in the station-line.

The principal objects for which the additional accuracy given by oblique offsets is desirable, are corners and intersections of boundaries, angles of buildings, mile-posts, and the like. When the object is a corner of a building, such as D in fig. 5, it is convenient to make each of the offsets, if possible (or at all events one of them), lie in a straight line with a face of the building, and so to determine the direction of such face or faces.

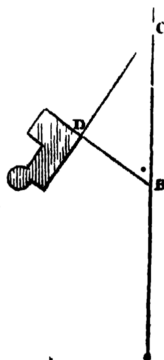


Fig. 5.

No general rules can be laid down for surveying the details of an intricate building, except that in many cases a rectangle may be set out so as to enclose it, and the sides of that rectangle used as station-lines from which to take offsets to the faces and corners of the building. To survey some buildings completely it is necessary to have access to the inside.

26. **Chained Triangles.**—It has already been stated in Articles

9 and 12, that the relative positions of different station-lines, and of the stations which they connect, are determined by so arranging them as to form a complete network of triangles over the district surveyed. In the absence of angular instruments, the figure of each of those triangles must be determined by measuring with the chain the length of each of its sides.

In fig. 6, let A B represent a station-line whose length and position are known; C, a third station lying out of the line. Then by measuring the two remaining sides, A C, B C, of the triangle A B C, so that the lengths of all its three sides may be known, the position of C is determined.

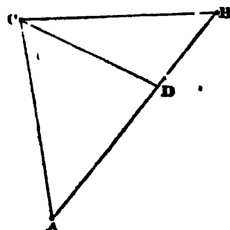


Fig. 6.

Agreeably to the principle already noted in the last article, that determination is the more accurate the less the angle A C B differs from a right angle. Supposing a certain error to have been committed in measuring one of the lines B C or A C, the consequent error in finding the position of C is equal to the original error if A C B is a right angle; but if that angle is either acute or obtuse, the error in the position of C is greater than the original error in the proportion of the *cosecant* of the angle A C B to radius.

Triangles in which the angle at the point to be determined is less than 30°, ~~or more than 150°~~, are said to be "*ill-conditioned*," and are avoided by skilful surveyors. In an ill-conditioned triangle, the error in the position of C is more than double of the corresponding error in the measurement of a side of the triangle.

The accuracy of the measurements in every important triangle should be checked by measuring a "*tie-line*," from one of its angles to a known point in the opposite side, such as C D in fig. 6. The agreement of the length of that line with the result of the measurements of the sides may be tested on the plan when plotted. It may also be tested by calculation; for if all the measurements are correct, the following equation will be verified,

$$CD^2 = \frac{AC^2 \cdot BD + BC^2 \cdot AD}{AB} - AD \cdot DB \ldots\ldots(1).$$

27. **Gaps in Station-Lines.**—A long station-line, otherwise well adapted for its purpose, may have one or more places in its course through which, owing to the intervention of buildings, woods, precipices, water, swamp, or other obstacles, it may be difficult or impossible to chain along the line with accuracy; and in some cases also it may be impossible to range the line directly across the obstacle. These difficulties are most readily met by the use of

angular instruments; but in the absence of such instruments, the chain alone may be used, according to methods which may be varied to suit the circumstances of each particular case.

Three kinds of cases may be distinguished:—*First*, those in which the obstacle can be seen over from side to side, and chained round, but not chained across. *Secondly*, those in which it can neither be seen over nor chained across, but can be chained round; and *Thirdly*, those in which the obstacle can be seen over, but neither be chained across nor chained round.

In each of the figures that illustrate this article, the inaccessible part of the station-line is marked by dots, and the direction in which the measurement proceeds is indicated by an arrow.

Case I.—*When the obstacle can be seen over*, the first operation is to plant a ranging pole in the station-line at the further side of the obstacle; and the problem to be solved is, to find the distance to that pole from some point already chained to on the nearer side.

First Method (By a parallel line, see fig. 7).—Let A and D be marks at the nearer and further sides of the obstacle respectively. By the optical square or otherwise, range A B, D C, at right angles to the station-line; make these perpendiculars equal to each other, and of any length that may be requisite in order to chain past the obstacle along B C, which will be parallel and equal to A D, the distance required; that is to say,

$$A\,D = B\,C \;\ldots\ldots\ldots\ldots\ldots\ldots \;(1.)$$

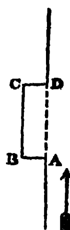


Fig. 7.

Second Method (By a triangle, see fig. 8).—A and D as before being points in the station-line at the nearer and further sides of the obstacle, set out a triangle A B C of any form and size that will conveniently enclose the obstacle, subject only to the conditions, that B and C are to be ranged in one straight line with D, and that the angles at B and C are neither to be very acute nor very obtuse. Measure with the chain the lengths A B, A C, B D, D C. Then the inaccessible distance A D is given by the formula.

$$AD = \sqrt{\left\{\frac{AB^2 \cdot CD + AC^2 \cdot BD}{BC} - BD \cdot CD\right\}}\,;\ (2.)$$

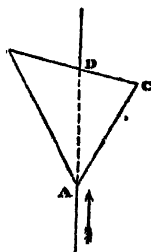


Fig. 8.

the computation of which will be much facilitated by the use of a table of squares.

That distance may also be found by plotting the triangle and the point D in its base on a sufficiently large scale, and measuring A D on the paper.

The figure of the obstacle may be surveyed by offsets from the sides of the triangle.

THIRD METHOD (By two triangles, see fig. 9). — Let b and c be points in the station-line at the nearer and further side of the obstacle respectively. From a convenient station A, chain the lines A b, A c, being two sides of the triangle A b c; connect those lines by a line B C in any position which will form a well-conditioned triangle A B C, of as large a size as is practicable: measure its three sides. Then the inaccessible distance is given by the formula,

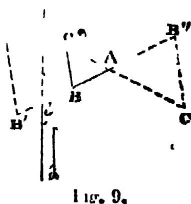


Fig. 9.

$$bc=\sqrt{\left\{Ab^2+Ac^2-\frac{Ab\cdot Ac}{AB\cdot AC}(AB^2+AC^2-BC^2)\right\}} \quad (3.)$$

The following modification of this formula, though less simple in appearance, is better adapted to computation by the help of a table of squares;

$$bc=\sqrt{\left\{Ab^2+Ac^2-\frac{(Ab+Ac)^2-(Ab-Ac)^2}{(AB+AC)^2-(AB-AC)^2}\cdot(AB^2+AC^2-BC^2)\right\}} \quad \ldots\ldots\ldots\ldots\ldots\ldots(3A.)$$

The points B and C are shown in the first instance as lying between A and the station-line; but if necessary, they may be taken in the prolongations of A c and A b beyond the station-line, as at B′ and C′, or in their prolongations beyond A, as at B″ and C″, and the same formula will still apply.

The formula is much simplified if A B and A C can be laid off so as to be respectively proportional to A b and A c; for then the triangles A B C and A b c become similar, B C is parallel to b c, and the inaccessible distance is simply

$$bc=BC\cdot\frac{Ab}{AB} \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (4.)$$

In this method, as well as in the preceding, the inaccessible distance may be found by plotting.

CASE II.—*When the obstacle can be chained round, but not chained across nor seen over.*

FIRST METHOD (By parallel lines, see fig. 10). — From A and

B, two points in the station-line on the nearer side of the obstacle, and at least as far apart as the distance across it is judged to be, lay off, by the optical square or otherwise, the equal perpendiculars A C, B D, of length sufficient to enable a straight line C D E F, parallel to the station-line, to be ranged and chained past the obstacle. Commence the chaining of this parallel line at D, in continuation of that of the station-line at B. As soon as the obstacle is passed, lay off the perpendicular E G equal to A C and B D; then G will be a point in the station-line beyond the obstacle, and the inaccessible distance will be

$$B G \quad D E \quad\ldots\ldots\ldots\ldots\ldots\ldots (5)$$

By continuing the parallel line and repeating the same process, additional points in the station-line, such as H, may be found.

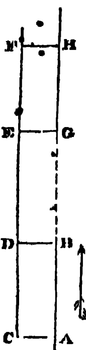


Fig. 10.

Second Method (By similar triangles, see fig. 11).—From a point A, as far back as practicable from the end B of the chained station-line on the nearer side of the obstacle, range two diverging lines A F, A E, past the two sides of the obstacle, in which measure the distances A D, A C, of two points D and C, which lie in one straight line with B. Continue the chaining of A F and A E, and make those distances respectively proportional to A D and A C, so that A D C and A F E may be similar triangles. Measure D C, in which note the position of B. Measure E F, in which take the point G, dividing E F in the same ratio in which B divides C D; then G will be a point in the station-line beyond the obstacle; and points still further on may be found, if necessary, by a similar process.

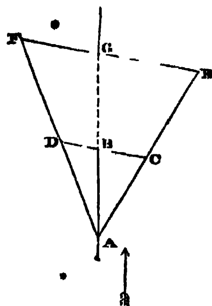


Fig. 11.

The inaccessible distance B G is found by the formula,

$$BG = \frac{AB \cdot CE}{AC}. \qquad (6.)$$

The boundaries of the obstacle can be surveyed by offsets from the sides of the quadrilateral C D F E.

Third Method (By transversals, see figs. 12, 13).—Let *a* and *b* be two points in the chained station-line at the near side of the obstacle, about as far apart as the inaccessible distance *b c* is judged to be. Mark a station C so as to form a well-conditioned

triangle with a and b; prolong the lines b C and a C until two points A and B are reached through which a straight line can be ranged and chained past the further side of the obstacle.

In some cases it may be advisable to begin by choosing the stations A and B, then to choose C, and then to range the lines B C a, and A C b (as in fig. 12), or A b C (as in fig. 13).

Fig. 12. Fig. 13.

All the sides of the two triangles A B C, a b C, are to be measured.

Then, *to find the point* c *at the intersection of the station-line with* A B, compute the distance of that point from B by one or other of the following formulæ:—

If c lies in A B produced, as in fig. 12

$$B c = \frac{A B \cdot a B \cdot b C}{C a \cdot A b - a B \cdot b C} \text{ .... } (7.)$$

If c lies between A and B, as in fig. 13,

$$B c = \frac{A B \cdot a B \cdot b C}{C a \cdot A b + a B \cdot b C} \text{ .... } (7 \text{ A.})$$

Next, *to find the inaccessible distance b c*, use the following formula (which is applicable to both figures):—

$$b c = \frac{a b \cdot A b \cdot B C}{C A \cdot a B - A b \cdot B C} \quad (8.)$$

The same problems may also be solved by plotting the figure a b c A B C a, and producing a b till it cuts A B, as in fig. 13, or A B produced, as in fig. 12. In a purely mathematical point of view, it is unnecessary to measure both A B and a b, as either of

those lines might be calculated from the other; but both should nevertheless be chained, as a check on possible errors.*

Case III. —*When the obstacle can be seen over, but neither chained across nor chained round.* This is the case of a station-line interrupted by a deep ravine, or a deep and rapid river. The first operation, as in Case I, is to range and fix a pole at *c* (fig. 14) in the station-line beyond the obstacle. The next is to find the distance *b c*.

First Method (By transversals).—On the nearer side of the obstacle, range the stations A and B in a straight line with *c*, making the angle *b c* B greater than 30°, and place them so that the intersecting lines A *b*, B *a*, connecting them with two points *a* and *b* in the station-line, shall form a pair of well-conditioned triangles *a b* C, A B C, as in the last problem. Measure the sides of these triangles, and compute the inaccessible distance *b c* by equation 8, already given.

Fig. 14.

As a check upon the position thus found for the point *c*, compute also the inaccessible distance B *c* by means of equation 7.

This problem is solved graphically by plotting the figure *a b c* A B C *a*, and producing *a b* and A B till they intersect in *c*.†

Second Method (By the optical square, when the inaccessible distance does not much exceed three or four chains, see fig. 15). B D being the inaccessible distance, at B, with the optical square, set out B C perpendicular to the station-line, and of a length such as to make B C D a well-conditioned triangle. At C, with the optical square, range C A perpendicular to C D, cutting the station-line in A. Measure A B, B C; then

$$\text{B D} = \frac{\text{B C}^2}{\text{A B}} \quad \ldots\ldots\ldots\ldots\ldots\ldots \quad (9.)$$

Fig. 15.

* The following are the formulæ for calculating A B from *a b*:—

In Fig. 12; $\text{A B} = \sqrt{\left\{\text{B C}^2 + \text{C A}^2 - \frac{\text{B C}}{b\,\text{C}}\,\frac{\text{C A}}{\text{C}\,a}\,(b\,\text{C}^2 + \text{C}\,a^2 - a\,b^2)\right\}}$.

In Fig. 13; $\text{A B} = \sqrt{\left\{\text{B C}^2 + \text{C A}^2 + \frac{\text{B C} \cdot \text{C A}}{b\,\text{C} \cdot \text{C}\,a} \cdot (b\,\text{C}^2 + \text{C}\,a^2 - a\,b^2)\right\}}$.

To compute *a b* from A B, interchange the positions of A and *a*, B and *b*, throughout the above formulæ.

† The solutions of this and the preceding problem are founded on the first theorem in Carnot's celebrated essay *On the Theory of Transversals*; a branch of Geometry at once simple in its principles and useful in its applications, but little known or studied.

The calculation represented by the formula 7, when each of the given distances is expressed by four figures, has been found to occupy about five minutes.

Methods for measuring gaps in station-lines by the aid of angular instruments will be explained in Chapter III.

28. **Field-Book.**—The writing and sketching in field-books is made either with ink or with an indelible pencil. If the book can be protected from rain, ink is to be preferred.

The field-book of a survey should commence with a sketch showing the general arrangement of the stations and station-lines relatively to the more conspicuous objects on the ground to be surveyed, made by the surveyor when he explores the country, as mentioned under head (*a*) of Article 12, page 12. Those stations may be distinguished by letters or by numbers. Principal stations are usually marked thus △. The remainder of the book will contain the detailed notes of the distances chained along the several station-lines, and the offsets measured from them.

In order that forward and backward, right and left, on the ground, may be represented by forward and backward, right and left, in the book, the successive notes written on each page begin at the bottom and proceed towards the top; and the pages are numbered from right to left. In the middle of each page is a vertical column broad enough to contain numbers of five or six figures. That column represents the station-line.

The surveyor begins at the bottom of the first page, by writing in the central column a letter, or other mark, to denote the station at which the line about to be chained commences, and beside it, a note stating between what stations the line runs: for example, "from A to B." As the chaining advances, he notes in the central column, proceeding upwards, the distances at which the station-line crosses boundaries, and traverses intermediate stations, and at which offsets are taken. Each distance of an intermediate station from the commencement is distinguished by enclosing it in an oblong or oval, and writing opposite to it the designation of the station, together with a reference to the other pages of the field-book in which the same station is referred to, and a note of its position upon other station-lines which traverse it. To the right and left of the central column are written the offsets measured to the right and left respectively, each opposite the figures denoting its distance from the commencement of the line; and those offsets are accompanied by a sketch-plan of the objects to which they are measured, with explanatory notes when required.

On arriving at the end of a station-line, the relative direction of the next line chained may either be stated in words—as, "turn to the right," "turn to the left"—or indicated by symbols like the following: ↰ ↱. At the commencement of each new station-line will be stated the position of the point from which it starts upon a former station-line.

Oblique offsets, small triangles, measurements of buildings, and the like, are best recorded by sketching a diagram of the lines measured, and writing their lengths along them.

The preceding explanation shows the general principles according to which field-books of chained surveys are kept. The details vary very much in the practice of individual surveyors. It is to be recommended that every surveyor should keep his field-book so distinctly that it may be possible for a draughtsman to plot the survey from the field-book without receiving any explanation from the surveyor.

29. **Plotting a Chained Survey.**—In plotting a survey, great attention should be paid to the absolute flatness of the drawing-board or table on which the paper is to be strained or laid, and to the perfect straightness of the straight-edge by which station-lines are to be ruled.

If the plan is to be mounted on cloth, the paper should be mounted before the plan is plotted; otherwise the mounting will alter its dimensions. On the whole, it is better *not* to "strain" the paper on which a survey is plotted on a drawing-board, in the way practised for architectural and mechanical drawings; because, when the paper is cut away from the board, and so relieved from the strain, it will contract, and perhaps contract unequally in different directions.

Each day's work should be plotted as soon as possible after having been surveyed.

The scale according to which the survey is plotted should at once be drawn on the plan, when it will contract and expand along with the paper.

The plotting is commenced by marking with a needle or pricker a point to represent the first station; drawing a straight line through that point to represent the first station-line, and laying down on that line, with a pair of beam-compasses, the positions of the other stations which it traverses.

The operations which follow consist chiefly in plotting triangles, and plotting distances and offsets.

30. **Plotting Triangles.**—The great triangles, whose sides connect the principal stations, are to be first plotted: then the secondary triangles, until the whole network is completed. The operation of plotting a triangle whose three sides have been measured is as follows:—A and B, fig. 16, represent two stations already plotted; the distances A C, B C, of a third station from those stations are known. With these distances as radii, describe with the beam-compasses a pair of small circular arcs about A and B respectively; the intersection of those arcs

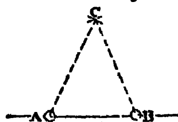


Fig. 16.

marks the required station C on the plan. Then with the straight-edge rule the lines A C, B C, and the triangle is complete.

It is usual, for the satisfaction of the engineer, and for future reference, to draw permanently on the plan, in a faint red colour, the principal station-lines, forming the primary network of triangles. Those lines are sometimes called "lines of construction." In some cases it is useful to draw permanently in the same way a portion of the secondary network of triangles: so far, at least as they can be used in computing areas.

When the plan of a survey extends over several sheets, it is necessary, in order to show the connection between two adjacent sheets, that a portion of at least one station-line, containing at least one principal station, should be plotted on each of the two sheets.

31. In **Plotting Distances, Offsets, and Details**, a flat ivory or box-wood scale is laid on the paper exactly parallel to the station-line, and loaded to keep it at rest: the divisions marked on its edge represent distances. A shorter flat scale, having broad ends exactly perpendicular to its edges, is laid on the paper with one end against the edge of the scale for distances: it is slid successively to the several distances from the station noted in the field-book, and the offsets are laid down by pricking with a needle opposite the proper graduations on one of its edges. Care should be taken that the offset-scale is exactly rectangular.

Oblique offsets are plotted like the sides of triangles.

In estate plans, on a large scale, different kinds of fences, such as stone-walls, hedges, palings, &c., are distinguished from each other by conventional modes of marking; but in plans for engineering projects, it is sufficient to distinguish between fenced and unfenced lines of division of land, marking the former by plain and the latter by dotted lines. In working plans on a large scale, walls may be shown of their proper thickness, and coloured red. Boundaries of parishes, counties, boroughs, and other legal divisions of the country, are marked with peculiarly shaped and arranged dots. Roads are coloured drab; streams and pieces of water light blue, with a darker shade along their edges. Dwelling-houses are coloured light red, out-buildings dark grey, public buildings light grey. In engraved plans buildings are shaded by diagonal hatching. Railways are marked by parallel lines representing rails; and in some cases these are crossed by short fine lines to indicate sleepers. Canals are distinguished from streams by their greater uniformity of width and regularity of course. Trees are indicated by sketching small figures somewhat resembling them.

There are conventional modes of indicating the nature of the surface of the ground, whether garden ground, arable land, pasture,

marsh, heath, and the like; but in plans for engineering projects it is sufficient to refer by numbers written on the plan to corresponding numbers in the book of reference, in which are stated the owner or reputed owner, lessee or reputed lessee, occupier, and description of each portion of property shown on the plan.

32. **Measuring Areas.**—The elementary methods of measuring areas which are useful in surveying are of three kinds:—the method of triangles,—the method of ordinates,—and the method by mechanism.

I. *Method of Triangles.*—Let a, b, c, denote the lengths of the sides of a triangle, and

$$s = \frac{a+b+c}{2},$$

the *half-sum* of those lengths; the area of the triangle is given by the formula—

$$\text{Area} = \sqrt{s(s-a)(s-b)(s-c)}; \dots\dots\dots\dots (1.)$$

or, using logarithms—

$$\text{log. area} = \frac{1}{2}\left\{\text{log. } s + \text{log. }(s-a) + \text{log. }(s-b) + \text{log. }(s-c)\right\} \quad (2.)$$

Another formula is as follows: let a be any one of the sides of a triangle; p the perpendicular upon that side from the opposite angle; then—

$$\text{Area} = \frac{a\,p}{2}. \dots\dots\dots\dots\dots\dots (3.)$$

Every right-lined figure can have its area calculated by dividing it into triangles, computing their areas by one or other of the preceding formulæ, and adding them together.

The areas of figures with curved outlines can be found approximately by this method, preceded by the process called "equalizing;" which consists in drawing through the curved boundaries a set of straight lines so as to enclose, as nearly as the eye can judge, the same area.

II. *The Method of Ordinates* is applicable to a long piece of ground of varying breadth, such as the stripe of land required for a railway, or the area represented in fig. 17. An axis is drawn along the greatest length of the figure; breadths are measured along ordinates at right angles to that axis, sufficiently close together to make the spaces between them approximate to trapezoids. Then let d be the distance along the axis between two adjacent ordinates, and b, b', the breadths of the figure at those ordinates; the area contained between that pair of ordinates is—

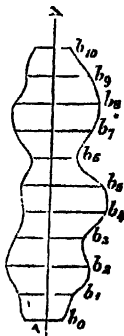


Fig. 17.

$$\frac{b+b'}{2}\cdot d;$$

and the area of the whole figure, being the sum of the areas of the parts into which it is divided by ordinates, is expressed as follows:—

$$\text{Area} = \Sigma \cdot \left(\frac{b+b'}{2}\cdot d\right);\ldots\ldots\ldots\ldots\ldots\ldots\ldots(4.)$$

Σ being a symbol of summation.

If the ordinates are at equal distances apart, all the values of d are equal, and the preceding formulæ becomes

$$\text{Area} = \left(\frac{b_o}{2} + b_1 + b_2 + b_3 + \&c. \ldots + \frac{b_n}{2}\right)\cdot d;\ldots\ldots\ldots\ldots(5.)$$

b_o and b_n being the breadths at the two ends of the figure, and b_1, b_2, &c., the intermediate breadths.

A modification of the last formula, founded on the assumption that the lateral boundaries of the figure consist of short parabolic arcs, is as follows, the number of divisions being even:—

$$\text{Area} = \left\{b_o + b_n + 2(b_2 + b_4 + \&c.\ldots) + 4(b_1 + b_3 + \&c.\ldots)\right\}\cdot\frac{d}{3}\ldots(6.)$$

The most accurate way to find the areas of all the pieces of land included in a survey, is to use the dimensions as given in the field-book alone, calculating the areas of the triangles by formula 1 or 2, and the areas of the stripes of land lying between the station-lines and the fences surveyed from them by formula 4, in which b and b' are to be taken to represent a pair of adjacent offsets, and d the distance between them.

This process, however, is very laborious, and may in many cases be dispensed with, by equalizing boundaries and taking measurements on the plan.

III. *Method by Mechanism.*—Instruments for measuring areas on plans by mechanism are called "Planimeters" and "Platometers;" and several have been contrived by different inventors; amongst others, General Morin and Mr. Sang.

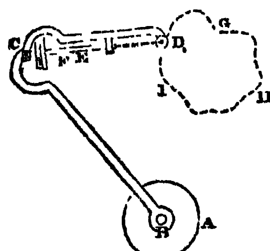


Fig. 18.

The simplest Planimeter is Amsler's, of which a sketch, showing its general principle, is given in fig. 18. A is a loaded disc which rests on the table, and serves as a fixed support for the instrument. In its centre, at B, is an upright pin, upon which turns the arm B C, to which at C is hinged the arm C D; so that the tracing

point at D can be moved in all directions over the paper. Exactly in the straight line C D is the axis E of the small wheel F, whose edge rests on the paper.

When the tracing point, D, is carried round the outline of any figure, such as G H I, so as to return finally to the point from which it started, it can be proved, that

$$\textit{Distance rolled by the edge of the wheel } \mathrm{F} = \frac{\textit{Area of Figure}}{\mathrm{C\,D}};$$

and consequently that

$$\text{Area of Figure} = \text{C D} \times \text{Distance rolled by the wheel F.}$$

C D is a measured constant length. The distance rolled by the wheel is measured by a graduated circle and vernier at one side of the wheel; the number of complete revolutions being recorded by another wheel, driven by an endless screw on the shaft E. This wheel and screw are omitted in the sketch. In Britain, the graduations on the circle usually represent square inches of area on the paper.

CHAPTER III.

OF SURVEYING BY ANGULAR MEASUREMENTS.

33. **Summary of Trigonometrical Formulæ used in Surveying.**

I. *Relations between Angles and Arcs.*—The *angle* or *difference of direction*, B A C, between two straight lines, A B, A C, which meet at the point A, is expressed as a quantity, as is well known, by stating how many of certain aliquot parts of a right angle it contains; those parts being, the *degree*, or ninetieth part of a right angle, the *minute*, or sixtieth part of a degree, the *second*, or sixtieth part of a minute, and the decimal fractions of a second. This mode of expressing angles is the most convenient for trigonometrical calculation. Another way of representing the same method is to conceive that a circle D E F is described about A in the plane of A B and A C with any radius; that the circumference of that circle is divided into 360 equal arcs called degrees, each degree into 60 minutes, each minute into 60 seconds, and so on: and that the number of such divisions of the circle contained in the arc D E which subtends the angle B A C is ascertained.

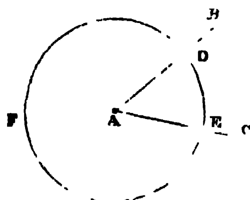


Fig. 19.

A second method of expressing the angle B A C is to take the ratio which the circular arc D E subtending it bears to the radius A D. In this case the angle is said to be expressed in terms of *arc to radius unity*, or in *circular measure*. This method, though less simple than the former, and less commonly employed, is useful in certain cases.

The two methods of expressing the same angle are compared with each other by the aid of our knowledge of the ratio which the circumference of a circle bears to its diameter: which ratio, although it cannot be expressed with absolute exactness by any number of arithmetical figures, can be calculated to any required degree of accuracy by successive approximations. It has been computed to about 250 places of decimals; but seven places of decimals are sufficient for ordinary purposes. The following table gives that ratio to eight places of decimals, with its common logarithm, and several ratios and logarithms deduced from it:—

	Ratios.	Logarithms.
Circumference of a circle to diameter 1; = Length of a semicircle to radius 1; = Area of a circle to radius $1 = \pi$;	3·14159265	0·4971499
Quadrant, or arc subtending a right angle, to radius 1;	$\frac{\pi}{2}$ = 1·57079632	0·1961199
Arc subtending one degree to radius 1;..............................	$= \frac{\pi}{180}$ = 0·01745329	8·2418774
Arc subtending one minute to radius 1;........................	0·0002908882	6·4637261
Arc subtending one second to radius 1;..............................	0·000004848137	4·6855749
Arc equal to radius, expressed in degrees,	57°·2957795	1·7581226
— — — in minutes,	3437′·747	3·5362739
— — — in seconds,	206264″·8	5·3144254
— — — in degrees, minutes, and seconds,	57° 17′ 44″	
— — — in decimal fractions of the circumference,	$\frac{1}{2\pi}$ = 0·1591551	9·2018201
Surface of a hemisphere to radius 1; ...	2π 6·2831853	0·7981799

The indices of the logarithms of fractions in the above table are affixed according to the system which is employed in trigonometrical calculations in order to avoid negative indices; that is to say, the index in each case is the complement of the proper negative index to 10, or the logarithm is that of the product of the fraction into 10,000,000,000.

The "centesimal" division of the quadrant into 100 degrees or "grades," 10,000 minutes, and 1,000,000 seconds is now nearly obsolete, even in France.

II. *Relations amongst Trigonometrical Functions of One Angle.*—The simplest mode of defining the trigonometrical functions of a given angle, such as the sine, cosine, &c., is to state that they are the ratios to each other of the sides of a right-angled triangle containing the given angle. Another mode, and the more common, is to state that they are represented by lines drawn in particular positions with respect to a circular arc of the radius unity, subtending the given angle.

In fig. 20, and also in fig. 21, A B, A C, are a pair of straight lines making with each other an *acute* angle, B A C.

In fig. 20, C is any point whatsoever in one of those lines, and C B a perpendicular let fall from that point upon the other line, so as to form a right-angled triangle, A B C.

In fig. 21, a circle of the radius unity is described about A, cutting off from each of the two lines a part equal to the radius, viz. :—

$$A\,D = A\,C = 1.$$

A F is a third radius, perpendicular to A D.

C B and C H are perpendiculars let fall from C upon A D and

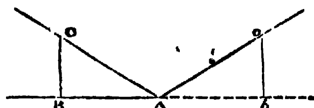


Fig. 20.

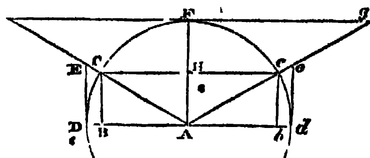


Fig. 21.

A F respectively; D E and F G are straight lines touching the circle at D and F (perpendicular, therefore, to A D and A F), and cutting A C produced in E and G respectively.

Then the definitions of the several trigonometrical functions of the angle B A C, according to the two methods, are as follows:—

	In Fig. 20.	In Fig. 21.
Sine, ...	$\frac{BC}{AC}$	..................B C = A H
Cosine, ...	$\frac{AB}{AC}$	....A B = C H
Versed Sine, ...	$\frac{AC-AB}{AC}$	....B D
Coversed Sine,	$\frac{AC-BC}{AC}$	....H F
Tangent, ...	$\frac{BC}{AB}$	....D E
Cotangent, ...	$\frac{AB}{BC}$	....F G
Secant, ...	$\frac{AC}{AB}$	....A E
Cosecant, ...	$\frac{AC}{BC}$	....A G

In fig. 20, the angles B A C and B C A are *complementary* to each other, being together equal to a right angle; so also are the angles B A C and C A F in fig. 21; and when this relation exists between a pair of angles, the sine of each is the cosine of the other, and so of all the other functions by pairs.

Denoting the angle B A C for brevity's sake by A, the following equations give the most important relations amongst its trigonometrical functions:—

$$\sin A = \sqrt{1 - \cos^2 A} = \frac{\tan A}{\sec. A} = \frac{1}{\operatorname{cosec} A}; \ldots\ldots (1.)$$

$$\cos A = \sqrt{1 - \sin^2 A} = \frac{\operatorname{cotan} A}{\operatorname{cosec} A} = \frac{1}{\sec A}; \ldots\ldots (2.)$$

$$\operatorname{versin} A = 1 - \cos A; \ldots\ldots\ldots\ldots (3.)$$

$$\operatorname{coversin} A = 1 - \sin A; \ldots\ldots\ldots\ldots (4.)$$

$$\tan A = \frac{\sin A}{\cos A} = \frac{1}{\operatorname{cotan} A} = \sin A \cdot \sec A = \sqrt{\sec^2 A - 1}; \ (5.)$$

$$\operatorname{cotan} A = \frac{\cos A}{\sin A} = \frac{1}{\tan A} = \cos A \cdot \operatorname{cosec} A = \sqrt{\operatorname{cosec}^2 A - 1}; (6.)$$

$$\sec A = \frac{1}{\cos A} = \sqrt{1 + \tan^2 A}; \ldots\ldots\ldots\ldots (7.)$$

$$\operatorname{cosec} A = \frac{1}{\sin A} = \sqrt{1 + \operatorname{cotan}^2 A}. \ldots\ldots\ldots\ldots (8.)$$

The trigonometrical functions of an obtuse angle are defined as follows:—

In fig. 20, and also in fig. 21, let A c be a straight line making with the line B A *produced beyond* A an angle b A c = B A C. Then the obtuse angle B A c is the *supplement* of the acute angle B A C, and is denoted by

$$180° - A.$$

From c in both figures let fall c b perpendicular to A b. In fig. 21, draw c H perpendicular to A F, and the tangents d e, F g, cutting A c produced in e and g.

Then in fig. 20, the right-angled triangle A b c is similar to A B C; and in fig. 21, the combination of lines on the right of A F is similar and equal to the combination of lines on the left; from which it appears, that all the trigonometrical functions of the obtuse angle 180° — A (with one exception to be presently pointed out), are equal in *numerical value* to the corresponding functions of its supplementary acute angle A. The one exception is the versed sine, which in fig. 21 is represented by D b = A D + A b = 2 A D — D B

In order the better to distinguish between trigonometrical functions of acute and obtuse angles, the principle is adopted, that inasmuch as A B and A b (in both figures) lie in *opposite directions* from A, they shall be regarded as having opposite signs:—that is, A B being positive, A b is negative; which amounts to laying down the rule, that *cosines of obtuse angles are negative.* The following are the relations between the trigonometrical functions of an obtuse angle $180° - A$, and its supplementary acute angle A, which arise from that rule:—

$$\left.\begin{array}{l}
\sin(180° - A) = \sin A; \\
\cos(180° - A) = -\cos A; \\
\text{versin}(180° - A) = 1 + \cos A = 2 - \text{versin}\ A; \\
\text{coversin}(180° - A) = \text{coversin}\ A; \\
\tan(180° - A) = -\tan A; \\
\text{cotan}(180° - A) = -\text{cotan}\ A; \\
\sec(180° - A) = -\sec A; \\
\text{cosec}(180° - A) = \text{cosec}\ A.
\end{array}\right\} \quad (9.)$$

From these equations it is to be understood, that in applying to obtuse angles trigonometrical formulæ which were originally intended for acute angles, the algebraical signs of all sines and cosecants of such angles are to be kept unchanged, and those of cosines, tangents, cotangents, and secants reversed.

In analytical geometry a further distinction is drawn between the *sines* of angles, whether acute or obtuse, lying to the right and left of a fixed direction, which are regarded as positive and negative respectively. In geodesy it is unnecessary to introduce that distinction, except in one case, to be explained afterwards.

III.—*Trigonometrical Functions of Two Angles.*

$$\sin A = 2 \sin \frac{A}{2} \cdot \cos \frac{A}{2} = \frac{2}{\text{cotan} \frac{A}{2} + \tan \frac{A}{2}} = \sqrt{\frac{1 - \cos 2A}{2}}. \quad (10.)$$

$$\left.\begin{array}{l}
\cos A = \cos^2 \frac{A}{2} - \sin^2 \frac{A}{2} = 1 - 2 \sin^2 \frac{A}{2} = 2 \cos^2 \frac{A}{2} - 1 = \\
\qquad \frac{\text{cotan} \frac{A}{2} - \tan \frac{A}{2}}{\text{cotan} \frac{A}{2} + \tan \frac{A}{2}} = \sqrt{\frac{1 + \cos 2A}{2}}
\end{array}\right\} \quad (11.)$$

$$\tan A = \frac{2}{\cot\frac{A}{2} - \tan\frac{A}{2}} = \sqrt{\frac{1 - \cos 2A}{1 + \cos 2A}}$$

$$= \frac{\sin 2A}{1 + \cos 2A} = \frac{1 - \cos 2A}{\sin 2A}. \quad (12.)$$

Let A and B be any two angles.

$$\sin (A + B) = \sin A \cos B + \cos A \sin B; \quad (13.)$$

$$\sin (A - B) = \sin A \cos B - \cos A \sin B; \quad (14.)$$

$$\cos (A + B) = \cos A \cos B - \sin A \sin B; \quad (15.)$$

$$\cos (A - B) = \cos A \cos B + \sin A \sin B; \quad (16.)$$

$$\tan (A + B) = \frac{\tan A + \tan B}{1 - \tan A \tan B}; \quad (17.)$$

$$\tan (A - B) = \frac{\tan A - \tan B}{1 + \tan A \tan B}; \quad (18.)$$

$$\sin A + \sin B = 2 \sin \frac{A+B}{2} \cdot \cos \frac{A-B}{2}; \quad (19.)$$

$$\sin A - \sin B = 2 \sin \frac{A-B}{2} \cdot \cos \frac{A+B}{2}; \quad (20.)$$

$$\cos A + \cos B = 2 \cos \frac{A+B}{2} \cdot \cos \frac{A-B}{2}; \quad (21.)$$

$$\cos B - \cos A = 2 \sin \frac{A-B}{2} \cdot \sin \frac{A+B}{2}. \quad (22.)$$

$$\tan A + \tan B = \frac{\sin (A + B)}{\cos A \cdot \cos B}; \quad (23.)$$

$$\tan A - \tan B = \frac{\sin (A - B)}{\cos A \cos B}, \quad (24.)$$

$$\cot A + \cot B = \frac{\sin (A + B)}{\sin A \cdot \sin B}; \quad (25.)$$

$$\cot B - \cot A = \frac{\sin (A - B)}{\sin A \sin B}; \quad (26.)$$

$$\sin^2 A - \sin^2 B = \cos^2 B - \cos^2 A = \sin(A - B) \cdot \sin(A + B); \quad (27.)$$

$$\cos^2 A - \sin^2 B = \cos^2 B - \sin^2 A = \cos(A - B) \cos(A + B); \quad (28.)$$

$$\tan^2 A - \tan^2 B = \frac{\sin(A - B)\sin(A + B)}{\cos^2 A \cdot \cos^2 B}; \quad (29.)$$

$$\cot^2 B - \cot^2 A = \frac{\sin(A - B) \cdot \sin(A + B)}{\sin^2 A \cdot \sin^2 B}; \quad (30.)$$

IV. *Formulæ for the Solution of Plane Triangles.*—All these formulæ are deduced from the two following principles:—

The sum of the three angles of a plane triangle is equal to two right angles.

The sides of a plane triangle are proportional to the sines of the opposite angles.

When the computations are to be made without the aid of logarithms, the simplest formulæ are the best; but when logarithms are used, formulæ of greater complexity are often employed, in order, as far as possible, to dispense with additions and subtractions, and make the calculation consist of multiplications and divisions.

Fig. 22 represents a plane triangle, whose three angles are denoted by A, B, C, and the three sides respectively opposite them by a, b, c.

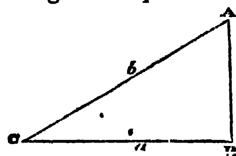


Fig. 22.

The following equations express in various forms the relation between the three angles, and enable this problem to be solved, *given, two of the angles, or trigonometrical functions of them: to find the third angle, or a trigonometrical function of it.*

$$A + B + C = 180°; \quad (31.)$$

Let A and B be given; then

$$C = 180° - A - B; \quad (32.)$$

$$\sin C = \sin(A + B); \ \cos C = -\cos(A + B); \quad (33.)$$

$$\tan C = \frac{\tan A + \tan B}{\tan A \tan B - 1}; \ \tan\frac{C}{2} = \cot\frac{A + B}{2}. \quad (34.)$$

Problem First.—When the Angles and one Side are given, let a be the given side; then the other two sides are

$$b = a \cdot \frac{\sin B}{\sin A}; \quad c = a \cdot \frac{\sin C}{\sin A}; \quad \ldots\ldots\ldots\ldots\ldots \text{(35.)}$$

or by logarithms,

$$\left.\begin{aligned} \log b &= \log a + \log \sin B - \log \sin A; \\ \log c &= \log a + \log \sin C - \log \sin A. \end{aligned}\right\} \quad \ldots\ldots \text{(35 A.)}$$

Problem Second.—When Two Sides and the Included Angle are given, let a, b, be the given sides, C the given included angle; then

1. *To find the third side*, the simplest formula is,

$$c = \sqrt{(a^2 + b^2 - 2\,a\,b \cos C)}; \quad \ldots\ldots\ldots\ldots\ldots \text{(36.)}$$

(observing, that if C is obtuse, the third term within the brackets is to be added instead of subtracted).

But this formula being unsuitable for logarithmic calculation, one or other of the following processes is substituted for it.

First Method.—

$$\text{make } \sin D = \frac{2\sqrt{a\,b}}{a + b} \cdot \cos \frac{C}{2}; \text{ then}$$

$$c = (a + b) \cos D \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots \text{(37.)}$$

Second Method:—

$$\text{make } \tan E = \frac{2\sqrt{a\,b}}{a - b} \cdot \sin \frac{C}{2}; \text{ then}$$

$$c = (a - b) \sec E. \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots \text{(38.)}$$

2. *To find the remaining angles*, A and B.

If the third side has been computed,

$$\sin A = \frac{a}{c} \cdot \sin C; \quad \sin B = \frac{b}{c} \cdot \sin C \quad \ldots\ldots\ldots\ldots \text{(39.)}$$

If the third side has not been computed,

$$\left.\begin{aligned} &\tan \frac{A+B}{2} = \text{cotan} \frac{C}{2}; \; \tan \frac{A-B}{2} = \frac{a-b}{a+b} \text{cotan} \frac{C}{2}; \\ &A = \frac{A+B}{2} + \frac{A-B}{2}; \; B = \frac{A+B}{2} - \frac{A-B}{2}. \end{aligned}\right\} \text{(40.)}$$

PROBLEM THIRD.—**When the Three Sides are given.**

To find any one of the angles, such as C, the simplest formula is the following:—

$$\cos C = \frac{a^2 + b^2 - c^2}{2\,a\,b}; \quad \ldots\ldots\ldots\ldots\ldots\ldots (41.)$$

but this formula being unsuited for logarithmic calculation, one or other of the four following formulæ is employed instead when logarithms are used. Let the half-sum of the sides of the triangle be denoted by

$$s = \frac{a + b + c}{2};\ \text{then}$$

$$\left.\begin{aligned} \cos\frac{C}{2} &= \sqrt{\frac{s\,(s-c)}{a\,b}}; \quad \sin\frac{C}{2} = \sqrt{\frac{(s-a)\,(s-b)}{a\,b}}. \\ \text{cotan}\,\frac{C}{2} &= \sqrt{\frac{s\cdot(s-c)}{(s-a)\,(s-b)}}; \quad \tan\frac{C}{2} = \sqrt{\frac{(s-a)\,(s-b)}{s\,(s-c)}} \end{aligned}\right\} \quad (42.)$$

When $\frac{C}{2}$ is a large angle, the expressions for $\cos\frac{C}{2}$ and $\text{cotan}\,\frac{C}{2}$ are the most convenient in calculation; when it is a small angle, those for $\sin\frac{C}{2}$ and $\tan\frac{C}{2}$ are to be preferred. A fifth formula, less used than the preceding, is

$$\sin C = \frac{2\sqrt{s\,(s-a)\,(s-b)\,(s-c)}}{a\,b}; \quad \ldots\ldots\ldots\ldots (43)$$

but this is unsuitable if C is nearly a right angle.

PROBLEM FOURTH.—**Two Sides given, and the Angle opposite one of them.**—In fig. 23, let A be the given angle, and a, c, the given sides, of which a is opposite A. The sine of the angle opposite c is given by the expression,

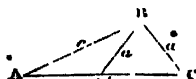


Fig. 23.

$$\frac{c}{a}\cdot\sin A; \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots (44.)$$

but this may apply either to the acute angle C or to its supplement, the obtuse angle C′ = 180° − C; C and C′ being the two points in the straight line A C′ C which are at the distance a from B. Unless, therefore, it is known by observation whether the angle opposite the side c is acute or obtuse, the solution of the problem is ambigu-

ous. Should that, however, be known, the angles can be computed, and thence the remaining side, by the method of the first problem. In general, problems that fall under the fourth case ought to be avoided in surveying, especially when the angle opposite c is nearly a right angle.

In all trigonometrical problems, it is to be borne in mind, that small acute angles, and large obtuse angles, are most accurately determined by means of their *sines*, *tangents*, and *cosecants*, and angles approaching a right angle by their *cosines*, *cotangents*, and *secants*.

PROBLEM FIFTH.—**To solve a Right-angled Triangle.**—All the preceding formulæ are applicable to this case; but they become very much simplified owing to the values assumed by the trigonometrical functions of the right angle, viz.:—

Sin $90° = 1$; cos $90° = 0$; tan $90°$ infinite; cotan $90° = 0$; sec $90°$ infinite; cosec $90° = 1$.

Let C denote the right angle; c the hypothenuse; A and B the two oblique angles; a and b the sides respectively opposite them. Then A and B are *complementary angles*, and the sine of each is the cosine of the other, as explained under Head II. of this article. The following cases may be distinguished:—

1. Given, the right angle, another angle B, the hypothenuse c. Then

$$A = 90° - B;\ a = c \cdot \cos B;\ b = c \sin B. \quad \ldots\ldots\ldots (45.)$$

2. Given, the right angle, another angle B, a side a,

$$A = 90° - B;\ b = a \cdot \tan B;\ c = a \cdot \sec B \quad \ldots\ldots (46.)$$

3. Given, the right angle, and the sides a, b,

$$\tan A = \frac{a}{b};\ \tan B = \frac{b}{a};\ c = \sqrt{a^2 + b^2} \quad \ldots\ldots\ldots (47.)$$

4. Given, the right angle, the hypothenuse c; a side a,

$$\sin A = \cos B = \frac{a}{c};\ b = \sqrt{c^2 - a^2}. \quad \ldots\ldots\ldots\ldots (48.)$$

5. Given, the three sides a, b, c, which fulfilling the equation $c^2 = a^2 + b^2$, the triangle is known to be right-angled at C.

$$\sin A = \frac{a}{c};\ \sin B = \frac{b}{c}. \quad \ldots\ldots\ldots\ldots\ldots\ldots (49.)$$

Problem Sixth.—To express the area of a plane triangle in terms of its sides and angles.

Case 1. Given, one side, c, and the angles.

$$\text{Area} = \frac{c^2}{2} \cdot \frac{\sin A \sin B}{\sin C} \quad \text{.................. (50.)}$$

Case 2. Given two sides, b, c, and the included angle A.

$$\text{Area} = \frac{b\,c \cdot \sin A}{2} \quad \text{...................... (51.)}$$

Case 3. Given, the three sides. See Article 32, page 33.

V. *Formulæ for the Solution of Spherical Triangles.*

These formulæ are all consequences of the two following principles :—

The sum of the three angles of a spherical triangle exceeds two right angles by an angle which bears the same proportion to four right angles that the area of the triangle bears to the surface of the hemisphere.

The sines of the angles of a spherical triangle are proportional to the sines of the angles subtended at the centre of the sphere by the sides to which they are respectively opposite.

Problem First.—To compute approximately the angles subtended by arcs on the earth's surface, and *vice versâ.*

In this calculation it is sufficiently accurate for the purposes of engineering geodesy to treat the earth's surface as a sphere of the diameter stated in Article 3, p. 2, viz. :—

41,778,000 feet = 7912½ statute miles;

so that, referring to the present Article, Division I., p. 37, for the proportions borne to the radius by arcs subtending various units of angle, we find, for the mean lengths of such arcs on great circles of the earth's surface, the following values,—*

* If it is desired to compute the lengths of small arcs on great circles somewhat more precisely, the following formulæ may be used.—

Let δ denote the difference between the latitude of the place and 45°, the sign + or — indicating whether that latitude is greater or less than 45°. Then the length in feet of an arc of the meridian which subtends one minute is

$$m = 6076 \cdot 36 \left(1 \pm \frac{\sin 2\delta}{200}\right); \quad \text{...................... (1.)}$$

	Mean length in Feet.	Logarithm.
Arc equal to radius,..................	20,889,000	7·3199176
Arc subtending one degree,.........	364,582	5·5617950
Arc subtending one minute,........	6076·36	3·7836437
Arc subtending one second,.........	101·273	2·0054925

Hence, let a be the length in feet of an arc on the earth's surface; α, the angle subtended by it in seconds; then

$$\alpha = a \div 101{\cdot}273 \text{ nearly. } \ldots\ldots\ldots\ldots\ldots\ldots \text{(52.)}$$

PROBLEM SECOND.—To compute approximately the *sines* of the angles subtended by small arcs on the earth's surface, and *vice versâ*.

Let $\frac{a}{r}$ be the ratio of a small arc a to the earth's radius r; α the angle subtended by it. Then it is known that the two following

the length in feet of an arc subtending one minute, on a great circle perpendicular to the meridian, is

$$m' = 6076{\cdot}86 \left(1 + \frac{1}{300} \pm \frac{\sin 2\delta}{600}\right); \ldots\ldots\ldots\ldots\ldots\ldots \text{(2.)}$$

and the length in feet of an arc subtending one minute on a great circle which makes an angle θ with the meridian, is

$$m'' = m \cos^2\theta + m' \sin^2\theta. \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots \text{(3.)}$$

The mean length of all the arcs subtending one minute on great circles which can be drawn through a given point is

$$\frac{m + m'}{2} = 6076{\cdot}36 \left(1 + \frac{1}{600} \pm \frac{\sin 2\delta}{300}\right). \ldots\ldots\ldots\ldots\ldots\ldots \text{(4.)}$$

At the parallel of 30° of latitude, which divides the surface of the hemispheroid into two nearly equal parts, the factor of this expression within the brackets is reduced to unity, and the length of the arc to its mean value; and the area of the surface of the spheroid is almost exactly equal to that of a sphere of the radius of 20,889,000 feet, corresponding to that value of the arc. It is for these reasons that 6076·36 feet has been adopted in this work as the true mean length of a nautical mile, rather than the length of a minute of the equator.

The area in square feet of a square, each of whose sides subtends one minute at a given latitude, is

$$m' = (6076{\cdot}36)^2 \cdot \left(1 + \frac{1}{300} \pm \frac{\sin 2\delta}{150}\right) \text{nearly}; \ldots\ldots\ldots \text{(5.)}$$

and this quantity also has its mean value at the parallel of 30°; viz. $(6076{\cdot}36)^2$.

The length in feet of a minute of longitude is given by the formula

$$m''' = m' \cdot \cos \cdot \text{latitude} \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots \text{(6.)}$$

In the preceding formulæ, the figure of a level surface is treated as if it were an exact spheroid of revolution with the polar and equatorial diameters in the ratio of 599 to 601, and no account is taken of various irregularities in the form of that surface, whose existence has been proved, but which have not yet been reduced to any general principle.—(See "Addenda," p. 789.)

series give approximations to the value of each of those quantities in terms of the other;

$$\sin \alpha = \frac{a}{r} - \frac{1}{1 \cdot 2 \cdot 3} \cdot \frac{a^3}{r^3} + \frac{1}{1 \cdot 2 \cdot 3 \cdot 4 \cdot 5} \cdot \frac{a^5}{r^5} - \&c. \ldots\ldots \quad (53.)$$

$$\frac{a}{r} = \sin \alpha + \frac{1}{1 \cdot 2 \cdot 3} \cdot \sin^3 \alpha + \frac{1 \cdot 3}{2 \cdot 4 \cdot 5} \sin^5 \alpha + \&c. \ldots \quad (54.)$$

In most cases which occur in engineering geodesy, the first two terms of each of those series are sufficient, and they may be thus expressed :—

$$\sin \alpha = \frac{a}{r}\left(1 - \frac{a^2}{6r^2}\right) \text{nearly}; \ldots\ldots\ldots\ldots \quad (55.)$$

$$\frac{a}{r} = \sin \alpha \left(1 + \frac{\sin^2 \alpha}{6}\right) \text{nearly}. \ldots\ldots\ldots\ldots \quad (56.)$$

For logarithmic calculation the following approximate formulæ are convenient :—

$$\log \sin \alpha = \log \frac{a}{r} - \cdot 0723824 \frac{a^2}{r^2}$$

$$= \log a \text{ (in feet)} - 7 \cdot 3199176 - \cdot 0723824 \frac{a^2}{r^2}; \ldots \quad (55\text{ A.})$$

$$\log a \text{ (in feet)} = 7 \cdot 3199176 + \log \sin \alpha + \cdot 0723824 \sin^2 \alpha \quad (56\text{ A.})$$

($\cdot 0723824$ = modulus of the common logarithms ÷ 6.)

Problem Third.—Given, the area of a spherical triangle on the earth's surface; to find the excess of the sum of the three angles above two right angles (or as it is called, the "spherical excess").

Let S be the area of the triangle, r the earth's radius, X the spherical excess; then

$$X = 360^\circ \cdot \frac{S}{2 \pi r^2}$$

$$= \text{angle subtended by arc equal to radius} \cdot \frac{S}{r^2}; \ldots\ldots (57.)$$

that is to say—

$$X \text{ (in seconds)} = \frac{206264 \cdot 8 \text{ S (in square feet)}}{436,350,321,000,000}$$

$$= \frac{\text{S (in square feet)}}{2,115,500,000 \text{ nearly}}; \ldots\ldots\ldots\ldots\ldots (58.)$$

or by logarithms,

$$\log X \text{ (in seconds)} = \log S \text{ (in square feet)} - 9 \cdot 3254101 \ldots (58\text{ A.})$$

In stating rules for the solution of spherical triangles, the word "side" is used for brevity's sake, when "the angle subtended by a side at the centre of the sphere" is meant. In fig. 24, A, B, C are the three angles of a spherical triangle; *a*, *b*, *c*, the sides respectively opposite them. The angles subtended respectively by these sides, which angles are called "the sides" in stating the rules, will be denoted by α, β, γ.

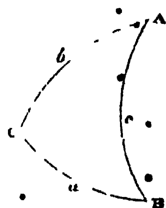


Fig. 24.

PROBLEM FOURTH.—Given, two angles of a spherical triangle, and the side between them; to find the remaining sides and angle—

Let A, B be the given angles, and γ the given side. Then to find the remaining sides α and β—

$$\tan\frac{\alpha+\beta}{2} = \tan\frac{\gamma}{2}\cdot\frac{\cos\frac{A-B}{2}}{\cos\frac{A+B}{2}};$$

$$\tan\frac{\alpha-\beta}{2} = \tan\frac{\gamma}{2}\cdot\frac{\sin\frac{A-B}{2}}{\sin\frac{A+B}{2}};\quad\ldots\ldots\ldots(59.)$$

$$\alpha = \frac{\alpha+\beta}{2}+\frac{\alpha-\beta}{2};\quad \beta = \frac{\alpha+\beta}{2}-\frac{\alpha-\beta}{2}.$$

To find the remaining angle C, we have the proportion—

$$\sin\alpha : \sin\beta : \sin\gamma :: \sin A : \sin B : \sin C \ldots\ldots\ldots(60.)$$

PROBLEM FIFTH.—Given, two sides of a spherical triangle and the angle between them; to find the remaining side and angle—

Let α, β be the given sides; C, the given angle.

First Method.—To find the remaining side γ;

$$\cos\gamma = \cos\alpha\cdot\cos\beta + \sin\alpha\cdot\sin\beta\cdot\cos C;\ \ldots\ldots\ (61.)$$

but this formula being unsuited to calculation by logarithms, the following has been deduced from it;

$$\text{Make } \sin D = \cos\frac{C}{2}\cdot\sqrt{\sin\alpha\cdot\sin\beta};\ \text{then}$$

$$\sin\frac{\gamma}{2} = \sqrt{\left\{\sin\left(\frac{\alpha+\beta}{2}+D\right)\cdot\sin\left(\frac{\alpha+\beta}{2}-D\right)\right\}};\ (62.)$$

and to find the remaining angles, we have the proportion,

$$\sin\gamma : \sin\alpha : \sin\beta :: \sin C : \sin A : \sin B.\ \ldots\ldots(63.)$$

Second Method.—To find the remaining angles, A, B.

$$\tan\frac{A+B}{2} = \frac{\cos\frac{\alpha-\beta}{2}\cdot\operatorname{cotan}\frac{C}{2}}{\cos\frac{\alpha+\beta}{2}};$$

$$\tan\frac{A-B}{2} = \frac{\sin\frac{\alpha-\beta}{2}\cdot\operatorname{cotan}\frac{C}{2}}{\sin\frac{\alpha+\beta}{2}}; \qquad \ldots\ldots(64.)$$

$$A = \frac{A+B}{2} + \frac{A-B}{2};\quad B = \frac{A+B}{2} - \frac{A-B}{2}.$$

The remaining side γ is found by the proportion (63).

PROBLEM SIXTH.—The three sides of a spherical triangle being given; to find the angles.

Let C be the angle sought in the first instance. Then

$$\cos C = \frac{\cos\gamma - \cos\alpha\cdot\cos\beta}{\sin\alpha\cdot\sin\beta}; \quad \ldots\ldots \qquad (65.)$$

but as this formula is not adapted for logarithmic calculation, one or other of the following, which are deduced from it, is to be employed for that purpose.—

Let $\sigma = \frac{\alpha+\beta+\gamma}{2}$ denote the half sum of the sides;

$$\cos\frac{C}{2} = \sqrt{\frac{\sin\sigma\cdot\sin(\sigma-\gamma)}{\sin\alpha\cdot\sin\beta}};\quad \sin\frac{C}{2} = \sqrt{\frac{\sin(\sigma-\alpha)(\sin\sigma-\beta)}{\sin\alpha\cdot\sin\beta}}.(66.)$$

$\cos\frac{C}{2}$ is best when $\frac{C}{2}$ approaches a right angle; $\sin\frac{C}{2}$ when $\frac{C}{2}$ is small.

These formulæ will serve alike to compute any angle. If it is desired to express the angle sought by A or by B, the following substitutions are to be made in the formulæ:—

For the following symbols in the formulæ for C, ...	α	β	γ
Substitute respectively in the formulæ for A, ...	β	γ	α
— — — — for B, ..	γ	α	β

PROBLEM SEVENTH.—In a right-angled spherical triangle, the right angle and any two other parts being given, to find the remaining parts.

Let C be the right angle, and γ the side opposite to it.

CASE 1.—Two sides being given, the third is found by the equation—

$$\cos\alpha\cdot\cos\beta = \cos\gamma; \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots(67.)$$

and the oblique angles by the equations—

$$\cos A = \text{cotan}\,\gamma \cdot \tan\beta;\ \cos B = \text{cotan}\,\gamma \cdot \tan\alpha;\ \ldots(68.)$$

or by the equations—

$$\text{cotan}\,A = \text{cotan}\,\alpha \cdot \sin\beta;\ \text{cotan}\,B = \text{cotan}\,\beta \cdot \sin\alpha.\ldots(69.)$$

Case 2.—Given, a side (α) and the opposite angle (A). Find the side β by the formula—

$$\sin\beta = \tan\alpha \cdot \text{cotan}\,A;\ \ldots\ldots\ldots\ldots\ldots\ldots(70.)$$

then find γ by (67) and B by (68) or (69).

Case 3.—Given, a side (α) and the adjacent angle (B). Find the side γ by the formula—

$$\text{cotan}\,\gamma = \cos A \cdot \text{cotan}\,\beta;\ \ldots\ldots\ldots\ldots\ldots\ldots(71.)$$

then find α by (67) and B by (68) or (69).

Case 4.—Given, two angles, A, B—

$$\cos\alpha = \frac{\cos A}{\sin B};\ \cos\beta = \frac{\cos B}{\sin A};\ \cos\gamma = \text{cotan}\,A \cdot \text{cotan}\,B.\ (72.)$$

VI. *Approximate Solutions of Spherical Triangles, used in Trigonometrical Surveying.*

As the largest triangles formed in trigonometrical surveying do not measure more than 100 miles in the side, and the ordinary triangles much less, the curvature of the arcs forming their sides is very slight, and their areas are very small fractions of that of a hemisphere of the earth; and consequently approximate methods of calculation can be applied to them, by which much of the labour is saved that would be required by a strict adherence to the rules of spherical trigonometry.

PROBLEM FIRST. Given, in a triangle on the earth's surface the length of one side, c, and the adjacent angles, A, B; to find approximately the third angle, C.

Calculate, by equation 50, p. 46, the *approximate area* of the triangle, as if it were plane. From that area, by equation 58, or 58 A, p. 48, calculate the "spherical excess" X. Then

$$C = 180° + X - A - B.\ \ldots\ldots\ldots\ldots\ldots\ (73.)$$

PROBLEM SECOND.—To find approximately the remaining sides, a, b, of the same triangle. Let α, β, γ be the angles subtended by the sides.

Method First (By spherical trigonometry).—Find the arc γ subtended by the given side c by equation 52, p. 47; or else find $\sin\gamma$ directly from c by equation 55 or 55 A, p. 48. Then

$$\sin\alpha = \frac{\sin A \sin\gamma}{\sin C};\ \sin\beta = \frac{\sin B \sin\gamma}{\sin C};\ \ldots\ldots\ldots(74.)$$

from which find α and β in seconds; then the lengths of the sides in feet are—

$$a = 101 \cdot 273\ \alpha;\ b = 101 \cdot 273\ \beta; \ldots\ldots\ldots\ldots(75.)$$

or a and b may be calculated directly from sin α and sin β by equation 56 or 56 A, p. 48.

Method Second (By plane trigonometry).—From each of the angles subtract one third of the spherical excess, and then treat the triangle as if it were plane. That is to say—

$$a = c \cdot \frac{\sin\left(A - \frac{X}{3}\right)}{\sin\left(C - \frac{X}{3}\right)};\ b = c \cdot \frac{\sin\left(B - \frac{X}{3}\right)}{\sin\left(C - \frac{X}{3}\right)}. \ldots\ldots(76.)$$

PROBLEM THIRD.—Given, in a triangle on the earth's surface, two sides a, b, and the included angle C; to find the remaining side, c, and angles, A, B.

Method First (By spherical trigonometry).—As in the last problem, find the angles α, β, subtended by the sides, by means of equation 52, p. 47, or the sines of those angles by means of equation 56 A, p. 48. Then solve the triangle as a spherical triangle, by means of equations 62 and 63, p. 49, or equation 64, p. 50. Lastly, make

$$c \text{ in feet} = 101 \cdot 273\ \gamma \text{ in seconds.} \ldots\ldots\ldots\ldots (77.)$$

Method Second (By plane trigonometry).—Compute the *approximate area* by equation 51, p. 46, as if the triangle were plane; thence compute the spherical excess X by equation 58 or 58 A, p. 48, and deduct one-third of it from the given angle. Then consider the triangle as a plane triangle, in which are given the two sides a, b, and the included angle $C' = C - \frac{X}{3}$. Find the third side c by equation 37, or equation 38, p. 43; and the remaining angles A′, B′, of the supposed plane triangle, by the equations 39 or 40, p. 43; and for the remaining angles of the real spherical triangle, take

$$A = A' + \frac{X}{3};\ B = B' + \frac{X}{3}. \ldots\ldots\ldots\ldots\ldots(78.)$$

PROBLEM FOURTH.—To diminish as far as possible the effects of small errors in angular measurements.

Such small errors are detected by measuring the whole three angles of a spherical triangle, and adding them together. If the measurements are perfectly correct, we shall find

$$A + B + C = 180° + X,$$

(X being the spherical excess, if it is appreciable). But if small

errors have been committed in measuring the angles, we shall find

$$A' + B' + C' = 180° + X \pm E,$$

where E is the total error. Then for the most probable values of the corrected angles are to be taken

$$A = A' \mp \frac{E}{3};\ B = B' \mp \frac{E}{3};\ C = C' \mp \frac{E}{3} \quad \ldots\ldots(72.)$$

The correction of each angle being one-third of the total error, and opposite in sign.

34. The **Theodolite** is an instrument whose chief use is to measure angles in a horizontal plane, or "*azimuths*," and which is occasionally used to measure also vertical angles, or *altitudes and depressions.*

When the word *Azimuth* is used without qualification, it usually means the number of degrees, minutes, and seconds by which the direction of a vertical plane passing through a station and a given object deviates to the *right* of a vertical plane passing through the station and the North Pole. When "Magnetic Azimuth" is specified, the angular deviation is reckoned from the magnetic meridian instead of the true meridian.

But the relative azimuth of any two objects may be measured at a given station; that is to say, the number of degrees, minutes, and seconds by which a vertical plane traversing the station and one of the objects deviates to the right of a vertical plane passing through the station and the other object.

An azimuth exceeding 180° denotes that the direction of the object to which it is measured lies to the *left* of the direction from which azimuths are measured, by an angle equal to the difference between the azimuth and 360°.

For example: in fig. 25, let A denote a station; A B the direction of the pole, or, as the case may be, of the object from which azimuths are measured, and which is held to have the azimuth 0°. C being an object which lies to the right of A B, its azimuth, being the number of degrees, &c., subtended by the arc *b c*, is equal to the angle B A C. On the other hand, D being an object lying to the left of A B, its azimuth, being the number of degrees, &c., subtended by the arc *b' d*, is equal to the difference between the angle B A D and 360°.

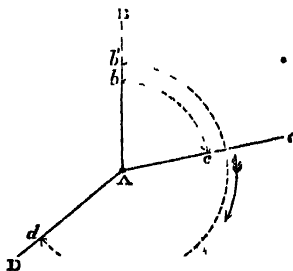


Fig. 25.

The horizontal angle between any two directions is the difference of their azimuths, if that difference is less than 180°; if it is greater than 180°, the excess of 360° above the difference of the azimuths is the angle between the directions

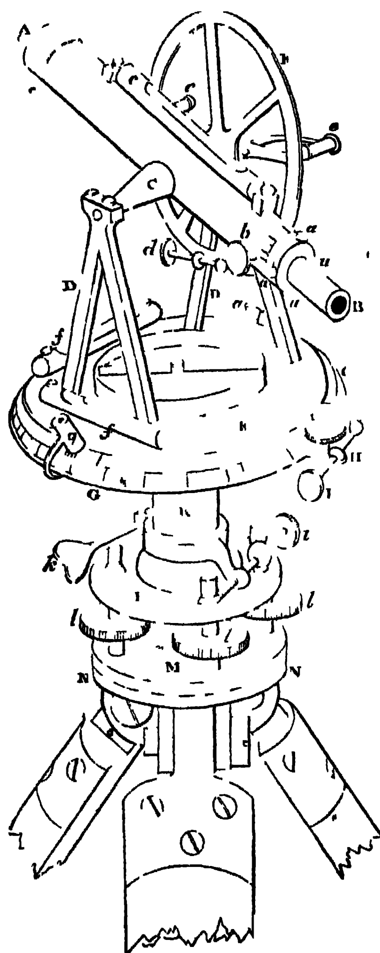


Fig 26.

Altitudes and depressions are the angles, always acute, which the directions of objects, as seen from a given station, make above and below a horizontal plane. The use of these angles will be further explained under the head of Levelling

The structure of theodolites varies very much; but there are certain essential parts which are common to all, and which will now be enumerated, commencing at the top, as they are found in the "Transit Theodolite" The usual material is brass, except for the "limb" or graduated ring of each of the circles, which is of silver or palladium

I. The *Telescope* A B consists of two tubes, one sliding within the other. The outer tube has, at its further end A, the object-glass, which forms at its focus an inverted image of the object looked at. The inner tube has, at its nearer end B, a combination of glasses called the "eye-piece," which magnifies that inverted image. By the use of an additional tube and certain additional glasses, an "erecting eye-piece" may be formed, which makes the object appear erect, but this causes loss of light, and possesses no particular

advantage. By moving the inner tube inwards and outwards by a rack and pinion, turned by the milled head *b*, the foci of the object-glass and eye-piece are adjusted till they coincide, which is known by the distinct and steady appearance of the image.

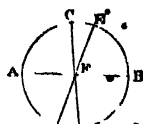


Fig. 27.

At the common focus of the object-glass and eye-piece, where the inverted image is seen, there is a "diaphragm" or partition, with a round hole in the middle crossed by three spider's lines, or equally fine platinum wires (see fig. 27); one horizontal, A B, and the other two, C D, E G, deviating slightly to opposite sides of a vertical plane. The point F where those wires cross each other should be exactly in the axis or "*line of collimation*" of the telescope; and the heads of four screws for adjusting it to that position are shown at *a*, *a*, *a*, *a*, in fig. 26.

II. The *Spirit-Level* *c* is attached to the outer telescope-tube by screws, by means of which it can be set exactly parallel to the line of collimation; so that when the air-bubble is in the centre of the level, the telescope is horizontal. The construction and use of spirit-levels will be further explained under the head of levelling.

III. The *Horizontal Axis* C, when the instrument is in adjustment, is exactly at right angles to the line of collimation, and exactly level; so that the telescope may turn about on the bearings of that axis in a truly vertical plane.

IV. The *Frames* or *Supports* (D, D) of the horizontal axis are high enough, in the transit theodolite, to admit of the telescope being turned completely over in a vertical plane; a motion which is useful in making certain observations. In Colonel Everest's theodolite the supports are made low, for the sake of compactness; but the telescope may be turned completely over when required, by lifting the horizontal axis out of its bearings. In the common theodolite the telescope is not fixed in the middle of that axis, but is supported in two forked rests called Y's, at the ends of a bar which is fixed at right angles to the horizontal axis; so that the telescope can be turned end for end by lifting it out of the Y's. When not required to be lifted out, the telescope is clasped firmly in its Y's by two semicircular arcs called "clips," which are hinged to the Y's at one side and fastened with pins at the other.

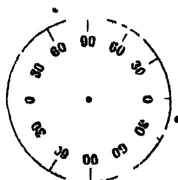


Fig. 28

V. The *Vertical Circle* or *Altitude Circle* E is fixed upon the horizontal axis. It is divided into four quadrants, the degrees in each of which are numbered from 0° to 90°, as indicated in the sketch, fig. 28. The two 0's are at the ends of the

diameter parallel to the line of collimation of the telescope; the two 90's at the ends of the diameter perpendicular to the former. There are two indices with verniers,* at opposite ends of a horizontal bar, read by the microscopes *e*, *e*; when the line of collimation is horizontal, each of those indices reads, or ought to read, 0°.

In directing the telescope to any object, it is turned at first by hand as nearly in the required direction as possible; then the vertical circle is "clamped," by turning a *clamp-screw* which lays hold of its lower edge; and then, by the *tangent-screw d*, a slow motion is given to the circle and telescope until the line of collimation points exactly towards the object.

In Colonel Everest's theodolite, instead of a complete vertical circle, there are two opposite sectors of about 90° each, so as to be capable of measuring altitudes and depressions as far as 45°; and the spirit-level is attached to the index-bar, instead of to the telescope.

In the common theodolite, instead of a vertical circle there is a semicircle only, having but one index and vernier.

VI. The *Vernier Plate* F (fig. 26), is a circular plate, fixed on the top of, and exactly perpendicular to, the *inner vertical axis* (concealed in the figure). It carries at its sides the supports D, D, of the horizontal axis, in its centre a *magnetic compass* with a glass top, and near its edge a pair of *spirit levels f*, *f*, at right angles to each other. At two points on its edge, diametrically opposite to each other, are two indices with verniers, read by means of the microscopes *g*, *g*. (In many theodolites there is but one microscope for this purpose, which is shifted round to the one or the other vernier as required.)

In Colonel Everest's theodolite the place of the lower horizontal circle is supplied by three horizontal arms diverging from the top of the inner vertical axis at equal angles of 120°, and having indices and verniers at their ends; and instead of the two spirit-levels *f*, *f*, there is one spirit-level fixed parallel to the horizontal axis.

VII. The *Horizontal Circle* G has its edge or limb bevelled to the figure of the frustum of a cone, and graduated; the degrees being numbered continuously round it towards the right, up to

* According to the ordinary construction of a vernier, its total length consists of a number of divisions of the primary scale less by one than the number of smaller divisions into which those divisions are to be subdivided. Suppose, for example, that the limb of one of the circles of a theodolite is divided to third parts of a degree, or 20′, and that it is to be subdivided by a vernier to third parts of a minute, or 20″, each subdivision being *one-sixtieth* part of a primary division: the length of the vernier will be 60 — 1 = 59 divisions of the primary scale, and it will be divided into 60 equal parts, each equal to 59-60ths of a division of the primary scale.

360°, as indicated by the sketch, fig. 29. The faces of the verniers are portions of the same conical surface. An arm projecting from the vernier-plate (or in Colonel Everest's theodolite, from the inner vertical axis) carries the *clamp* H for laying hold of the circle after the telescope has been turned approximately towards an object by hand, and the *tangent-screw* I for giving the vernier-plate a slow motion until the line of collimation points exactly towards the object.

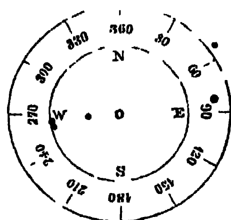


Fig. 29.

The size of the circles of a theodolite, both horizontal and vertical, and the minuteness of their graduations, depends on the extent and accuracy of the operations for which they are intended. Four inches and eight inches in diameter are about the extreme limits of diameter for horizontal circles in those made for any ordinary purpose, though a few have been made of larger sizes. Those most commonly used in surveying have circles of five inches in diameter, divided into half-degrees, and subdivided by the verniers to single minutes, and by estimation with the eye to half or quarter minutes. For such purposes as the principal triangulation of the survey for a line of railway, and for ranging curves, a larger theodolite is requisite: it is generally sufficient to use one with circles of six inches in diameter, divided to twenty minutes, and subdivided by the verniers to twenty seconds, and by estimation with the eye to ten seconds.

VIII. The *Outer Vertical Axis* K is fixed to the horizontal circle, and is a tube, containing within it and accurately fitting the inner vertical axis. It turns round on a ball-and-socket joint at its lower end; and is clamped in any required position by means of a collar with a tightening-screw *k*. From the collar projects an arm, acted upon by means of the tangent-screw *i*, so as to give a slow motion in azimuth to the horizontal circle when the outer vertical axis is clamped. The fixed nut of this screw is attached to—

IX. The *Upper Parallel Plate* L, through a cylindrical socket in which the outer vertical axis passes, so as to be always at right angles to it. The four *plate-screws l, l, l* (and a fourth concealed), serve to place the vertical axes truly vertical, by adjusting the position of the plate L relatively to—

X. The *Lower Parallel Plate* M, to the centre of which the outer vertical axis is attached by means of the ball-and-socket joint before mentioned. This plate is screwed upon—

XI. The *Staff-Head* N, which is supported by three strong wooden legs. In the middle of the lower side of the staff-head

directly under the vertical axis, is screwed a hook (concealed in the figure), from which a plummet is hung, in order to ascertain whether the centre of the theodolite is exactly over the station on the ground.

Instead of the upper parallel plate, Colonel Everest's theodolite has three diverging arms (fig. 30), as in an astronomical circle, with a vertical foot-screw supporting the end of each. The lower end of each screw has a shoulder, by means of which it is held down to the plate which forms the top of the staff-head; and those shoulders form the only attachment between the staff-head and the instrument. The chief advantage of this construction is, that the three foot-screws can be adjusted with one hand; whereas the adjustment of the four plate-screws in the ordinary construction requires both hands.

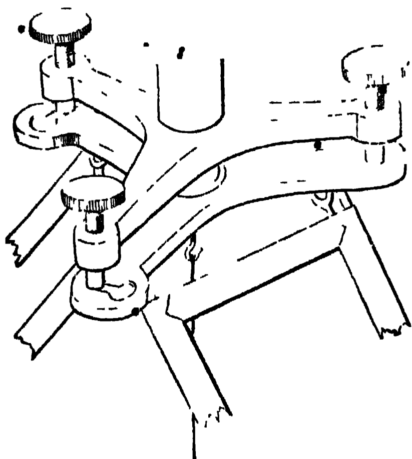


Fig. 30.

In some theodolites a second telescope is attached below the horizontal circle, in order, by directing that telescope on an object, to test whether the circle has been disturbed during the interval between two observations.

35. **Adjustments of the Theodolite.**—The adjustments of the theodolite, as well as those of every other surveying instrument, may be distinguished into temporary adjustments, which are made by the user of the instrument each time that it is set up, and permanent adjustments, which are made by the manufacturer, and only tested and corrected occasionally by the user.

I. The *Temporary Adjustments* will now be described, on the supposition that the permanent adjustments are correct.

(1.) Place the theodolite at the station by the aid of the plumb-line mentioned in Division XI. of the last article.

(2.) To "level the instrument"—that is, to place the vertical axis truly vertical—the easiest process is to make the vernier-plate truly horizontal by means of the spirit-levels *f*, *f*. For that purpose it is to be turned into such a position that the two spirit-levels *l*, *l* shall be parallel respectively to the two diagonals of the square formed by the plate-screws. Then the bubble is to be

brought to the centre of each level by turning the pair of plate-screws to whose plane the level lies parallel.

A more exact adjustment, however, can be made by means of the level *c* attached to the telescope, because it is larger and more delicate than those attached to the vernier plate. To effect this adjustment, turn the vernier plate till the telescope is over one pair of plate screws: by the aid of the tangent-screw *d*, adjust the vertical circle carefully to 0°; turn the pair of plate-screws under the telescope until the bubble is brought to the centre of the spirit-level: turn the vernier-plate round through 180°; if the bubble now deviates from the centre of the spirit-level, correct one-half of the deviation by the tangent screw *d*, and the other half by the plate-screws: turn the vernier-plate through 90°, so as to bring the telescope over the other pair of plate-screws, by means of which bring the bubble to the centre of the level again: the vertical axis is now truly vertical.

If the bubbles are not now at the centres of the vernier-plate levels *f*, *f*, those levels are not truly perpendicular to the vertical axis; but the correction of this error belongs to the permanent adjustments.

In Colonel Everest's theodolite the vertical axis is adjusted by means of the level which is parallel to the horizontal axis, by first placing that level parallel to a line joining any two of the three foot-screws, and bringing the bubble to the centre by turning one or both of them, and then turning the upper part of the instrument through 90°, and bringing the bubble to the centre of the level in its new position by means of the third foot screw.

(3.) To adjust the telescope for the prevention of "parallax"—that is, to bring the foci of the glasses to the cross-wires,—look through the telescope, and shift the eye-piece in and out until the cross-wires are seen with perfect distinctness. Then direct the telescope to some well-defined distant object, and by means of the milled head *b*, shift the inner tube in and out until the image of the object is seen sharp and clear, coinciding apparently with the cross-wires.

The latter part of this adjustment has to be made anew for each new object at a different distance from the preceding one. The nearer the object, the further must the inner tube be drawn out.

A good test of the adjustment for parallax is to move the head from side to side while looking through the telescope. If the adjustment is perfect, the image of the object will seem steadily to coincide with the cross-wires: if imperfect, the image will seem to waver as the head is moved. If the image seems to shift in the *opposite* direction to the head, the inner tube must be drawn out further; if in the *same* direction, it must be drawn inwards.

II. The *Permanent Adjustments* should be tested from time to time; but in a well-made theodolite they will seldom require correction. Before testing those adjustments, the temporary adjustments should be made with care.

(1.) The *Adjustment of the Line of Collimation*, in a transit theodolite, and also in Colonel Everest's, consists in placing that line precisely at right angles to the horizontal axis. To effect this, direct the line of collimation towards some very distinct distant object, bringing, by means of the tangent-screw of the horizontal circle, the cross-wires to coincide in azimuth with the image of a well defined point in that object. The vertical circle should be unclamped. Now lift the horizontal axis out of its bearings, and replace it with the ends reversed, so that the telescope is upside down; if the cross-wires now coincide in azimuth with the same object, the line of collimation is perpendicular to the horizontal axis; if not, one half of the deviation is to be corrected by shifting the cross-wires by means of the horizontal adjusting-screws of the diaphragm, and the other half by the tangent screw of the horizontal circle. Reverse the horizontal axis again, and repeat the operation till the adjustment is perfect.

In the transit theodolite there is another mode of reversing the telescope to perform this adjustment, which consists in turning the telescope over on its horizontal axis, and then turning it round through exactly 180° in azimuth.

In the common theodolite the line of collimation is adjusted by turning the telescope half round in its Y's about its own axis, and observing whether the cross-wires continue to coincide with the same object. Should they deviate, half the deviation is to be corrected by the diaphragm-screws, and the other half by the tangent screw of the horizontal circle. This adjustment places the line of collimation in coincidence with the axis of the Y's. The adjustment of the latter line perpendicular to the horizontal axis is left to the instrument maker.

(2.) The *Adjustment of the Level attached to the Telescope* can only be effected, in the transit theodolite, by methods which will be explained in treating of the adjustment of levelling instruments. The same may be said of the adjustment in a vertical direction of the line of collimation. (See Art. 50.)

In the common theodolite, having levelled the level attached to the telescope by the tangent-screw of the vertical circle, lift the telescope out of the Y's and set it down again turned end for end. If the bubble deviates from the centre of the level, correct half the error by the adjusting-screws which connect the level with the telescope, and the other half by the tangent-screw of the vertical circle.

(3) To ascertain the *Index-error of the Vertical Circle*, set the vertical axis truly vertical with great care, as described under the head of temporary adjustments; set the spirit-level on the telescope exactly level; observe the reading of the vertical circle; if it is 0°, there is no error; if it differs from 0°, the difference is an error in the position of the index of the vertical circle, to be allowed for in each angle measured.

(4.) The *Adjustment of the Horizontal Axis exactly perpendicular to the Vertical Axis* is generally left to the instrument maker; but in some theodolites there are adjusting-screws for the supports of the horizontal axis. In this case the perpendicularity of the horizontal to the vertical axis may be tested by directing the telescope on an object whose altitude is considerable; then turning it round through exactly 180° in azimuth, and turning it over in a vertical plane so as to look at the same object. If the cross-wires can again be brought to coincide with the object, the adjustment is correct; if not, half the deviation is to be corrected by the tangent-screw of the horizontal circle, and the remainder by the adjusting-screws of the supports; and the operation is to be repeated till the adjustment is found to be correct.

This adjustment may also be tested by observing whether, when the instrument is clamped in azimuth, the cross-wires traverse an object and its image as reflected from a level surface of fluid.

36. **Measuring Horizontal Angles with the Theodolite.**—To measure the horizontal projection of the angle subtended at a given station A, by the direction of two objects B and C,—in other words, the difference of azimuth of the two objects,—set up the theodolite at the station, and make the temporary adjustments as described in the preceding Article. The outer vertical axis being clamped, and the vernier-plate and vertical circle A unclamped, direct the telescope towards one of the objects (as B), as accurately as possible by the hand; clamp the vernier-plate, and by its tangent-screw bring the cross wires to cover the object exactly. Read the degrees, minutes, and seconds indicated by one vernier, and the minutes and seconds indicated by the other, and note them down. Find the mean arc indicated, by setting down the entire degrees as read on the first vernier, and the mean between the additional arcs in minutes and seconds as read by the two verniers.

Unclamp the vernier-plate, direct the telescope towards the other object (C), and proceed as before, taking care to read the entire degrees on the same vernier.

The difference between the mean arcs read off when the line of collimation is directed towards B and C respectively, is the required difference of azimuth, or the horizontal angle B A C.

The object of reading the minutes and seconds on both verniers,

and taking the mean, is to correct the effect of any errors which might arise from the vertical axis not being exactly concentric with the graduated limb of the horizontal circle. In fig. 31, let E C and D B be two straight lines cutting each other in A, a point not in the centre of the circle B C D E. The eccentricity of that point produces equal and opposite deviations in the arcs B C and D E from the arc which would subtend an angle equal to B A C at the centre of the circle; so that the mean of those arcs is exactly equal to the arc which correctly measures the angle B A C, how great soever the eccentricity may be.

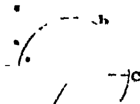


Fig. 31.

The same object is attained in Colonel Everest's theodolite by taking the mean of the arcs read off by the three equidistant verniers, which are used in order to give better security against errors in graduation than two verniers give.

In the transit theodolite, errors arising from the horizontal axis not being exactly perpendicular to the vertical axis may be eliminated by turning the telescope over about the horizontal axis, and half round about the vertical axis, repeating the measurement of the angle in this new position, and taking the mean of the results.

When a series of horizontal angles has been measured at a station between a series of objects, returning at last to the object with which the observations commenced, the accuracy of the observations may be tested by adding the angles together; when their sum ought to be exactly 360°. Should it differ by a small arc from 360°, the most probable values of the several angles will be found by dividing the total error by the number of errors to find the correction, which is to be added to or subtracted from each of them according as their sum is too small or too large.

When very great accuracy is required in measuring a horizontal angle, the effect of errors of graduation may be diminished to any required extent by the process called REPETITION, which is as follows:—

Clamp the *vernier-plate*, and read the verniers.

Unclamp the *vertical axis*: direct the telescope towards B; clamp the vertical axis, and direct the line of collimation exactly towards B by the tangent-screw of the *vertical axis.*

Unclamp the *vernier-plate*; direct the telescope towards C; clamp the vernier-plate, and direct the line of collimation exactly towards C by the tangent-screw of the *vernier-plate.*

Unclamp the *vertical axis*, &c. (as before).

Repeat the whole operation as many times as it is required to reduce the errors of graduation, observing always to direct the

line of collimation towards B by turning the vertical axis, and towards C by turning the vernier-plate. Finally, the line of collimation being pointed towards C, read the verniers, remembering to reckon 360° for each complete revolution of the vernier-plate upon the horizontal circle.

The difference between the arcs read at the beginning and at the end of the process will be equal to the arc subtended by the angle B A C multiplied by the number of *repetitions*; and being divided by that number will give the required angle. The multiplied arc will be affected by only one error of graduation, which will be divided in finding the required arc; so that the error in the final result will be equal to the original error divided by the number of repetitions.

This process diminishes the effect of errors of observation somewhat, but not in the same proportion with errors of graduation; because an observer tends in general to make errors in the same direction at each observation; and such errors accumulate by repetition.

37. **Reflecting Instruments** are used chiefly in navigation and marine surveying; but as they are occasionally used in land surveying also, a general description of their construction and action will be given here.

The principle upon which reflecting instruments act is this:—that if there are two plane mirrors whose reflecting surfaces make a given angle with each other, and a ray of light, in a plane perpendicular to the planes of both mirrors, is reflected from both successively, its direction after the second reflection makes with its original direction an angle which is double of the angle made by the mirrors with each other.

One application of this principle—the optical square—has already been described in Article 24, page 21.

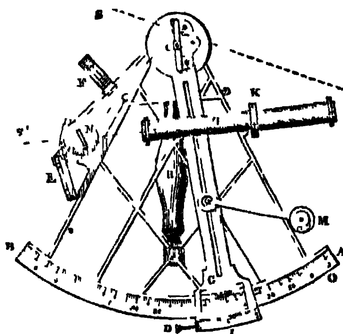


Fig. 32.

The **Sextant** (fig. 32) is of the form of a sector of a circle, of 60°, and sometimes rather more, in angular extent. A B is the graduated limb, on which the degrees are of one-half of the extent of those on a non-reflecting instrument; so that for example, an exact sextant is divided into 120 degrees instead of 60°. C G is the index, having a vernier, and a microscope M for reading the divisions. At the back of the

instrument is a clamp-screw, not shown, for holding the index in any required position, and at D is a tangent-screw for giving it a slow motion to complete its adjustment. The two mirrors have their planes at right angles to the plane of the instrument; one of them, called the "*index-glass*," C, is carried by the index at its centre of motion; the other, called the "*horizon-glass*," N, is carried by the frame of the sector; half of it is silvered and the remainder unsilvered. The unsilvered half is the further from the face of the instrument. Both mirrors should be made of strong plate glass, with its surfaces exactly plane and parallel.

T is a telescope directed towards the horizon-glass. It is carried by a ring K, and capable of adjustment to a greater or less distance from the plane of the instrument; and the object of that adjustment is, to vary the proportions of the light received from the silvered part and through the unsilvered part of the horizon glass, so as to render the images of two luminous objects seen directly and by reflection equally bright, although the objects themselves may be unequally bright. That equalization of brightness is favourable to accuracy in observing angles.

E and F are sets of darkening glasses, of various colours and shades, which are used when required, to moderate the light from very bright objects, such as the sun.

H is the handle by which the instrument is held.

Sextants for nautical purposes usually have the graduated limb of from six to eight inches radius, the graduated limb being subdivided by the vernier to 20′ in the smaller sizes, and 10″ in the larger. The observer can in each case read to one half of these arcs by estimation.

The nautical sextant is seldom used for land surveying. For that purpose the *box sextant* is employed, and for triangles of small extent only, not exceeding about a mile in length of side. The box-sextant is a sextant so small as to be entirely contained within a cylindrical brass box of about three inches in diameter and two inches in depth. It is graduated to half-degrees, and subdivided by the vernier to minutes, and by estimation to half-minutes. It is usually furnished with a small telescope, which, however, it is seldom necessary to employ, a plain sight-hole being used instead. The index is moved by a pinion and toothed sector.

The box-sextant has sometimes a contrivance added for enabling it to measure angles greater than 120°. That contrivance depends on the principle, *that if two reflected rays of light proceed in the same direction from two mirrors which make an angle with each other, the directions of the rays before reflection make double that angle with each other*. and it consists of a small mirror below the index-glass, fixed in such a position that when the index is at the mark num·

bered 180° upon what is called the "supplementary arc," those two mirrors are at right angles to each other; and the objects whose images as seen in them appear to coincide in direction, lie in fact in diametrically opposite directions.

Troughton's **Reflecting Circle** is an instrument having the mirrors and telescope of a sextant, together with a completely circular limb, and three indices radiating from its centre at angles of 120°. By observing each angle with the instrument in two positions, reading each angle observed upon the three verniers, and taking the mean of the six results, some of the errors of a sextant are avoided, and others diminished.

The **Universal Instrument,** as improved by Professor Piazzi Smyth, is a sort of reflecting circle, in which a spirit level with a very small bubble is so placed that by means of a lens and a totally reflecting prism an image of the bubble is formed at the focus of the telescope, and the coincidence of the centre of that image with the cross-wires shows when the line of collimation is truly horizontal.

The **Adjustments of the Sextant are as follows:—**

(1.) To place the index-glass exactly perpendicular to the plane of the instrument. This adjustment is made by the maker, but the observer may test it by setting the index to about 60°, and looking at the image of the limb of the instrument as reflected in the index-glass, when the real limb and the image ought to seem to form one continuous arc.

(2.) To place the horizon-glass exactly perpendicular to the plane of the instrument. This adjustment is tested by clamping the index near to 0°; looking at some well-defined far distant object, and turning the tangent-screw of the index till the object as seen directly and its reflected image are made to seem to coincide, if possible. If the horizon-glass is correctly adjusted, it will be possible to make the apparent coincidence exactly; if not, the glass must be corrected by means of adjusting screws with which it is fitted.

(3.) To ascertain the "*index error*," the angle marked by the index is to be read off when the above-mentioned coincidence has been made. If there is no index-error, the index will mark exactly 0°: any deviation from this is to be noted down as the index-error of the instrument, and allowed for in all future angular measurements. For the purpose of measuring the index-error when it is negative (that is—when the correction for it is to be added), the graduations of the limb are carried a short distance back from 0°. In reading this part of the limb (called the "arc of excess"), the divisions of the vernier are to be reckoned the reverse way.

(4.) The parallelism of the line of collimation of the telescope to the plane of the instrument is tested by placing the index so as to produce the apparent coincidence of two distinct objects whose

directions make an angle of 90°, or thereabouts, and observing whether a slight motion of the plane of the sextant about an axis traversing the object seen by reflection disturbs the apparent coincidence, which it should not do if the adjustment is correct.

38. **Use of the Sextant in Surveying.**—To measure a horizontal or nearly horizontal angle with the sextant, hold the instrument so that the plane of its face shall pass through the two objects subtending the angle: look through the telescope or sight-hole at the object which is farthest to the left, so as to see it through the unsilvered part of the horizon-glass; move the index by hand until the reflected image of the right-hand object is seen in the silvered part of the horizon-glass; clamp the index, and move it slowly by the tangent-screw till that image apparently coincides with the left-hand object. (In the box-sextant, the entire motion of the index is produced by turning the pinion.) Then read the angle by means of the index and vernier, and add or subtract the index-error according as it lies behind or in advance of 0°.

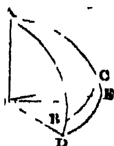
In fig. 32, P S′ represents the direction of the left-hand object; P S that of the right-hand object. When the image of the latter appears to coincide with the former, the rays of light coming from the right-hand object are reflected from the mirror C to the mirror N, and thence to the eye in the same direction with those which come directly from the left-hand object; and according to the principle stated at the beginning of the last article, the angle made by the directions of the objects S, P, S′, is double of that made by the planes of the mirrors. When the mirrors are parallel to each other, the index points to 0° (or deviates from that point by the index-error only); and the divisions marked as degrees on the limb are of half the length of actual degrees; so that the angle read off on the limb (index-error being allowed for), is the angle between the directions of the objects. If there is much difference in the distinctness of the objects, the less distinct object should be looked at directly; and should it lie to the right of the other, the face of the sextant must be turned downwards.

In order that the angle measured may be a horizontal angle, the two objects and the observer's eye must be at the same level. When this is not the case, three methods may be followed. The least accurate is, to choose by the eye two objects in the same vertical planes with the objects whose relative azimuth is to be found, and as nearly as possible on a level with the observer's eye, and to measure the angle between these. To attain greater accuracy, two vertical poles are to be ranged and adjusted by the plumb-line, in the directions of the two objects, and the angle between them measured with the plane of the sextant horizontal. In using the box-sextant for the details of a survey, one or other of these methods

is in general sufficiently accurate, if the ground is not very hilly.

The most accurate method is to measure the angle between the objects themselves, and to take also the angle of altitude or depression of each. (The taking of such angles will be further considered under the head of levelling.) The *zenith distance* of each object is found by subtracting its altitude from, or adding its depression to, 90°.

In fig. 33, let O represent the observer's station; O B, O C, the directions of the objects; B O C the angle between them; O D E a horizontal plane; D O B and E O C the altitudes of the objects; O A a vertical line, and A D E a spherical surface.


Fig. 33.

Then, in the spherical triangle A B C, the three sides are given, viz., A B and A C, the zenith distances, and B C, the angle between the objects; and the horizontal projection of that angle, being equal to the angle A, may be computed by the proper formula. (See Article 33, Division V., equation 66, p. 50.)

39. **Use of the Compass in Surveying.**—It has already been mentioned that the theodolite is usually provided with a compass, carried on the centre of the vernier plate. This compass consists of a magnetic needle, hung by an agate cap on a point in the centre of the instrument, and of a flat silver ring fixed round the inside of the compass-box, and divided into degrees and half-degrees, the numbering of which usually commences at a point exactly under the telescope, and proceeding *towards the left*, goes completely round the circle, ending at the point where it started, which is marked 360°. There is a small catch, by pressing which the magnetic needle is lifted off its bearing when not in use, to avoid unnecessary wear; and by which also its vibrations are gradually checked when an observation is made. To find the magnetic bearing of any object from a given station, the line of collimation of the telescope is directed towards it; and the surveyor, when the vibrations of the needle have ceased, reads the angle to which the north end of the needle points, and which denotes *so many degrees to the east of north.* When the angle to the east of north exceeds 90°, it is to be observed that 90° east of north means *east,* 180° east of north, *south,* and 270° east of north, *west.* In some cases, however, the ring is divided into four quadrants, the points in a line directly under the telescope being both marked 0°, and the points in a line perpendicular to the telescope, 90°, as in fig. 28, p. 55; and then the bearing is read so many degrees to the east of north, west of north, east of south, or west of south, as the case may be.

The compass most frequently used in surveying is the *Prismatic*

Compass, consisting of a glass-covered box three or four inches in diameter, in which is hung a magnetic needle: the needle carries a light graduated silver ring fixed upon it, and the box has sights fixed to its rim. The farther sight, when in use, stands upright to a height equal to the diameter of the box, and contains a vertical slit with a vertical wire in the middle. The near sight has a very small slit to look at the object through, below which is a totally reflecting magnifying prism, so placed as to show to the eye of the observer a reflected and magnified image of that part of the edge of the graduated ring which is directly below the line of sight. He directs the sight towards an object, and at the same time, and with the same eye, reads its bearing on the ring. In order to show bearings in degrees to the east of north, the numbering of the degrees on the ring begins at the south end of the needle, proceeds *towards the right*, and goes completely round to 360°.

The "*Circumferenter*" is a compass with sights mounted on a stand, chiefly used in surveys of mines.

The horizontal angle subtended by two objects may be found to a rough approximation by taking the difference of their magnetic bearings.

The compass cannot be read in surveying to less than a quarter of a degree; and considering the continual changes which go on in the earth's magnetism, and the effects of local attraction, it is thought doubtful by the best authorities whether magnetic bearings can be relied upon even to half a degree. Hence, although it is a convenient instrument for filling up small details, and making rough surveys, it is not to be used where accuracy is required.

It is usual to mark the magnetic north upon a plan, and this can easily be done by taking the magnetic bearing of one of the principal station-lines. The true north ought to be shown also, and the means of finding its direction will be explained in Article 42.

40. **Great Trigonometrical Survey.**—The general nature of a survey of this class has already been stated in Article 12, Division (*c*), p. 13, viz.: -measuring one base-line with extreme accuracy, and finding the lengths of all the other sides of triangles by calculation from their angles. A few sides of triangles may be measured in parts of the survey far distant from the original base, in order to test the accuracy of the whole work: these are called *bases of verification.*

The trigonometrical calculations required in a survey of this class consist almost entirely in computing the remaining sides of a triangle when one side and two of its angles are given: as to which computation, if the triangle is sensibly plane, see Article 33, Division IV., equation 35, p. 43, and if it is sensibly spherical, see Article 33, Division VI., pp. 51 to 53.

The following points require some further explanation:—

I. *Ill-conditioned Triangles*, that is, triangles with any angle of less than 30° ~~or more than 150°~~, are to be avoided in surveying by angles as well as in surveying by the chain, and for the same reason. (See Article 26, p. 24.)

II. *Checking Angles.*—The whole three angles of each great triangle should be measured, in order that the accuracy of the observations may be checked by adding them together, when they ought to amount to 180° (+ the spherical excess, if sensible; see Article 33, Division V., equation 58, p. 48). The treatment of unavoidable errors has been explained in Article 33, Division VI., Problem 4, p. 53.

The accuracy of the measurement of the internal angles of any polygon on the earth's surface may be checked by adding them together; when, if n denotes the number of the sides of the polygon, the angles ought to amount to

$(n-2)\ 180°$ + the spherical excess, calculated from the area of the figure as for a triangle.

III. *Checking Sides.*—In a complete network of triangles, it will always be found that many of the sides are so placed that their lengths can be calculated independently from different sets of data, which gives the means of checking the accuracy of the measurements and calculations.

IV. *Prolonging the Base.*—As it is necessary that the base should be measured on a level piece of ground, it is in general of limited extent, and much shorter than the sides of the great triangles; and its ends, also, are seldom in commanding positions suited for stations. Such a base line is "*prolonged*," by ranging straight lines in continuation of it, at one or both ends, until a sufficient length has been obtained and suitable stations reached, the length of such additional lines being computed from angular measurements, as follows:—In fig. 34, let A B be the measured base, and B E a line ranged in continuation of it. Choose a lateral station C, so that A B C and B C E shall be well-conditioned triangles; measure the three angles of each of these triangles; from the angles A C B, C A B, and the base A B, compute the side B C; and from that side, and the angles C E B, B C E, compute the additional length B E. Take another lateral station D, at the opposite side of the base, and by solving, in the same manner, the triangles A B D, D B E,

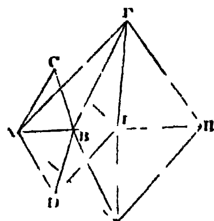


Fig 34

compute B E from independent data, so as to check the previous determination of its length.

E H represents a farther prolongation of the base line, and F and G the lateral stations which form the triangles by means of which its length is computed. At each of those stations angles are measured between all the previously determined points, A, B, E in order that there may be as many ways of verifying the calculations as possible. In the same manner the base may be prolonged either way as far as may be deemed necessary.

V. *Enlarging Triangles.*—A mode of connecting a comparatively short base with the sides of large triangles, without prolonging it, or introducing ill conditioned triangles, is as follows:—In fig. 35, let A B represent the base. Choose two stations C and D, at opposite sides of the base, and as far from each other as is consistent with making A C B and A D B well-conditioned triangles. From each of those four points measure the angles subtended by the other three. Then calculate the sides A C, C B, B D, D A; when there will be data for computing the length of C D in a variety of different ways, which will check each other. Taking C D as a new base, choose a pair of stations E and F still farther asunder, and proceed as before to determine the distance E F, and so on until a distance has been determined sufficiently long to serve as the side of a pair of triangles in the great triangulation.

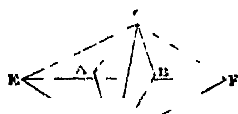


Fig. 35.

41. **Great Traversing Surveys.**—The general nature of a survey of this class, as usually required for a long line of communication, has been explained in Article 12, pp 12, 13, and illustrated by figure 3, p. 12. Some further explanation will now be given on the following points:—

1. *Checking Distances and Angles.*—The lateral objects, such as F, G, H, &c., in fig. 3, are generally inaccessible or unavailable as stations for the theodolite; so that the only angles measured for the main triangulation are those at the stations A, B, C, &c. If errors were impossible, the measurement of the base lines A B, B C, C D, &c, and of the angles between them, A B C, B C D, &c., would be sufficient to determine their lengths and directions. The use of the lateral objects is to check the results of those measurements, in the following manner:—

In the triangle A B F, the side A B, and the angles at A and B having been measured, calculate the side B F. In the triangle B F C, the side B F having been calculated, and the angles at B and C having been measured, calculate the side B C, the result being compared with the length of the same line as measured on

the ground, will check the accuracy of the work so far. The process of comparison is precisely similar for each successive main station-line of the survey.

II. *Gaps in the Main Station-lines,* such as have already been referred to in Article 27, are in most cases to be measured by the process already described in Article 40, Division IV, p. 69, and illustrated by fig. 34, for prolonging a base-line. In that figure A B may be held to represent a measured portion of the station-line, and B E, or B H, the gap or inaccessible distance. The sides of the lateral triangles formed in order to determine that distance may also be used as station-lines for the details of the survey.

Fig. 36 shows how a distance C D between two objects is to be measured, when both ends of it are inaccessible to chaining. Measure a base A B, having its ends so situated that the six lines connecting them and the objects C and D with each other may form well-conditioned triangles, and at the stations A and B measure the angles C A D, D A B, A B C, C B D. In the triangle C A B, compute the sides A C, B C; in the triangle D A B, compute the sides A D, B D. Then, in the triangle C A D, in which the sides A C and A D, and the included angle at A are given, compute the third side C D, as shown in Article 33, Division IV., Problem 2, equations 37, 38; also compute C D by the same process as the third side of the triangle C B D; the two results will check each other.

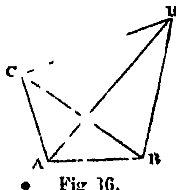


Fig 36.

42. **Finding the Meridian.**—For the purpose of laying down the direction of the true north on the plan of an engineering survey, the angle which one of the principal station-lines makes with the meridian must be determined, though not with the same accuracy that is required for astronomical and geographical purposes. The following are some of the methods:—

I. *By the Two greatest Elongations of a Circumpolar Star.*—This, the most accurate method, consists in observing the greatest and least horizontal angles made by a star near the pole with a station-line of the survey, when the star is at its greatest distances east and west of the pole, and taking the mean of those angles, which is the true azimuth of the station-line, or horizontal angle which it makes with the meridian. In the northern hemisphere the Pole-star, α Ursæ Minoris, is the best for this purpose.

This method, however, is seldom practicable with an ordinary theodolite, as in general, one of the observations must be made by daylight.

II. *By equal altitudes of a Star.*—The theodolite being at a station in the station-line chosen, measure the horizontal angle from

the station-line to any star which is not near the highest or lowest point of its apparent daily course, and take also the altitude of that star. Leave the vertical circle clamped, and let the instrument remain perfectly undisturbed until the star is approaching the same altitude at the other side of its apparent circular course. Then, without moving the vertical circle, direct the telescope towards the star, clamp the vernier-plate, and by the aid of its tangent-screw, follow the star in azimuth with the cross wires until it arrives exactly at its former altitude, as is shown by its image coinciding with the cross wires; then measure the horizontal angle between the new direction of the star and the station-line: the mean between the two horizontal angles will be the true azimuth of the station-line.*

In both the preceding processes it is to be understood that the *mean of two horizontal angles* means their *half-sum* when they are at the same side of the station line, but their *half-difference* when they are at opposite sides.

The second method may be applied to the sun, observing the sun's west limb in the forenoon and east limb in the afternoon, or *vice versâ;* but in that case a correction is required, owing to the sun's change of declination. When the sun's declination is changing towards the $\left\{\begin{matrix}\text{north}\\ \text{south}\end{matrix}\right\}$, the approximate direction of the meridian, as found by the method just described, is too far to the $\left\{\begin{matrix}\text{right}\\ \text{left}\end{matrix}\right\}$. The correction required is given by the formula,†

$$\frac{\text{change of sun's declination}}{2} \times \sec \ \text{latitude} \times \operatorname{cosec} \frac{1}{2} \ \text{angular motion of sun between the observations} \quad \ldots \ \ldots \ (1.)$$

III. *By One greatest Elongation of a Circumpolar Star.*—To use this method, the declination of the star, and the latitude of the place, should be known. Then

$$\sin \cdot \text{azimuth of star at greatest elongation} = \cos \cdot \text{declination} \div \cos \cdot \text{latitude}; \ \ldots\ldots\ldots\ldots\ldots \ (2.)$$

and this azimuth, being added to or subtracted from the horizontal angle between the station-line and the star when at its greatest elongation (according as the station-line lies to the same side of

* In observing at night with the theodolite, it is necessary to throw, by means of a lamp and a small mirror, enough of light into the tube to make the cross-wires visible.

† At the equinoxes, the rate of change of the sun's declination is about 59″ per hour; and it varies nearly as the cosine of the sun's right ascension.

the meridian with the star, or to the opposite side) gives the azimuth of the station-line.*

IV. *By observing the Altitude of a Star, and the Horizontal Angle between it and the Station-line.*—The altitude being corrected for refraction, the azimuth of the star is computed by taking the zenith-distance, or complement of that altitude, the polar distance† of the star, and the co-latitude of the place, as the three sides of

* The following is a table of the declinations of a few of the more conspicuous stars for the 1st of January, 1897, together with the annual rate at which those declinations are changing, + denoting increase, and — diminution:—

NORTHERN HEMISPHERE.

	Star.	North Declination.			Rate of Annual Variation
α	Andromedæ,	28°	31′	19″	+ 19″·9
α	Ursæ Minoris (Pole-Star),	88	45	30	+ 18 ·8
α	Arietis,	22	58	31	+ 17 ·2
α	Ceti,	3	41	8	+ 14 ·3
α	Persei,	49	29	41	+ 13 ·1
α	Tauri (Aldebaran),	16	18	9	+ 7 ·5
α	Aurigæ (Capella),	45	53	35	+ 4 ·0
α	Orionis (Betelgeuze),	7	23	16	+ 0 ·9
α	Geminorum (Castor),	32	6	52	— 7 ·6
α	Canis Minoris (Procyon),	5	29	21	— 9 ·0
β	Geminorum (Pollux),	28	16	29	— 8 ·5
α	Leonis (Regulus),	12	28	14	—17 ·5
α	Ursæ Majoris,	62	18	24	—19 ·4
η	Ursæ Majoris,	49	49	38	—18 ·1
α	Bootis (Arcturus),	19	43	7	—18 ·8
α	Ophiuchi,	12	38	6	— 2 ·8
α	Lyræ (Vega),	38	41	15	+ 3 ·2
α	Aquilæ (Altair),	8	35	46	+ 9 ·3
α	Cygni,	44	54	44	+ 12 ·7
α	Pegasi (Markab),	14	39	3	+ 19 ·3

SOUTHERN HEMISPHERE.

	Star.	South Declination			Rate of Annual Variation
β	Orionis (Rigel),	8°	19′	14″	— 4″·5
α	Columbæ,	34	7	44	— 2 ·1
α	Argûs (Canopus),	52	38	22	+ 1 ·9
α	Canis Majoris (Sirius),	16	34	32	+ 4 ·7
α	Hydræ,	8	12	44	+ 15 ·4
η	Argûs,	59	8	34	+ 18 ·8
α	Crucis,	62	31	41	+ 20 ·0
α	Virginis (Spica),	10	37	26	+ 18 ·9
α	Centauri,	60	24	27	+ 15 ·0
α	Scorpii (Antares),	26	12	12	+ 8 ·2
α	Trianguli Australis,	68	50	18	+ 7 ·1
α	Pavonis,	57	3	53	—11 ·2
α	Gruis,	47	27	35	—17 ·3
-	Piscis Australis (Fomalhaut),	30	10	5	—19 ·0

† The polar distance is the complement of the declination.

a spherical triangle; when the azimuth of the star will be the angle opposite the side representing the polar distance. (See Article 33, Division V., Problem 6, p. 50.) The azimuth of the station-line is then to be found as in Method III.

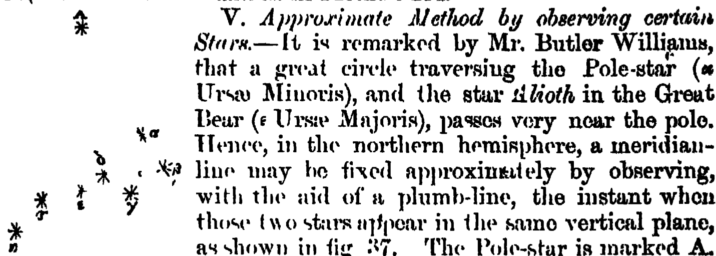


Fig 37.

V. *Approximate Method by observing certain Stars.*—It is remarked by Mr. Butler Williams, that a great circle traversing the Pole-star (α Ursæ Minoris), and the star *Alioth* in the Great Bear (ε Ursæ Majoris), passes very near the pole. Hence, in the northern hemisphere, a meridian-line may be fixed approximately by observing, with the aid of a plumb-line, the instant when those two stars appear in the same vertical plane, as shown in fig 37. The Pole-star is marked A.

When two points on the earth's surface have the same latitude, but different longitudes, the horizontal angle made by their meridians with each other is found by the following equation;

$$\sin \frac{1}{2} \text{ horizontal angle} = \sin \frac{1}{2} \text{ difference of long.} \times \sin \cdot \text{lat.} \quad (3.)$$

43. **Plotting and Protracting.**—The most accurate method of laying down the angles of great triangles on paper is to calculate the lengths of the sides of the triangles, and plot them with beam-compasses like chained triangles (Article 30, p. 31).

To plot, according to this principle, a solitary angle, like that between a station line and the meridian, a circle is to be drawn, with as large a radius as is practicable, round the station where the angle is to be laid down. Then the distance between the points where the two lines enclosing the angle cut that circle is found by multiplying the radius by the *chord* of the angle—that is, twice the sine of half the angle.

But to save time where less accuracy is required, especially in laying down secondary triangles and details, angles are laid down at once, or "protracted," by the aid of instruments called "protractors;" being flat graduated circles or parts of circles, which are laid on the paper. They are of various constructions and various degrees of accuracy.

The most accurate *Circular Protractor* has a round piece of plate-glass in its centre, through which the paper can be seen. The under side of the glass touches the paper, and has the centre of the graduated circle marked on it by a fine cross. The circle is divided to half-degrees, and subdivided to minutes by the vernier on its index. The index has two diametrically opposite arms, each of which has hinged on its end a branch carrying a pricker,

which is held up clear of the paper by a spring. When the index has been turned to any required degree and minute on the circle, the two branches are pressed down, and their prickers mark two points on the paper which are in the required direction, and which are or ought to be in one straight line traversing the centre of the circle. It is often convenient to draw, by the aid of those prickers, a graduated circle on the paper, through the centre of which lines making any required angle can be drawn, and their directions transferred, so as to pass through any required station on the paper, by the aid of a large and accurate parallel ruler.

The *Semicircular Protractor* has a straight side, which can be slid along a straight-edge fixed to the table or drawing board into any required position. Its index has a long arm projecting beyond the circle, with a straight fiducial edge, which is used to rule lines in any required direction through any station on the plan.

44. **Traversing on a Small Scale** has been referred to in Article 12, Division (*c*), p. 13, as a means of surveying long, narrow, and winding objects in detail. The most accurate way of performing it is to form a series of triangles by means of lateral objects, as already described in that article, and in Article 41, the checking of the accuracy of the work being tested by plotting, without calculation. Each lateral object is traversed by at least three lines from different stations in the survey; and those three or more lines will intersect each other in one point on the paper, if the station-lines between the stations and the angles at the stations have been correctly measured and plotted.

In almost all mining surveys, and in some above ground, it is impossible to take suitable angles to lateral objects, the only angles capable of being measured being those which the station-lines make with each other. In such cases the station-lines should be laid out so as to return to the starting point, and form a "closed polygon." The accuracy of measurement of the angles may then be tested by taking the sum of all the "salient" angles of that polygon—that is, of those which project outwards—and subtracting from it the sum of the "re-entering angles"—that is, of those which project inwards. The result (which is the *algebraical sum* of the angles of the polygon) ought to be

$$180^\circ \times \left\{ \text{number of salient angles} - 2 - \text{number of re-entering angles} \right\} \quad \ldots (1.)$$

Before plotting such a survey, the angle made by each station-line with one fixed direction ought to be computed (by successive additions or subtractions of the angles which those lines make with each other) and protracted on the paper by drawing a line to

represent that fixed direction, placing the zero-points of the protractor on that line, and laying off the directions of the several station-lines as described in Article 43. The accuracy of the measurement of distances, and of the plotting of distances and angles, is tested by the exactness with which the end of the last station-line on the paper coincides with the starting-point of the first.

In surveying by *traversing with the compass and chain*, the angles observed at each station are the directions which the station-lines that meet at it make with a fixed or nearly fixed direction, viz., that of the magnetic meridian. The zero-line on the paper, therefore, represents that meridian; and the angles protracted from it are simply the several magnetic bearings of the station-lines. Traversing with the compass and chain is accordingly an easy and rapid method of surveying; but as explained in Article 39, p. 67, its want of accuracy makes it suitable only for small or rough surveys.

45. **Plotting by Rectangular Co-ordinates**, or by **Northings, Southings, Eastings**, and **Westings**, is the most accurate way of plotting a traverse, because the position of each station is plotted independently, and not affected by the errors committed in plotting previous stations. It consists in assuming two fixed lines or *axes*, as O X and O Y, fig. 38, crossing each other at right angles at a fixed point O, computing the perpendicular distances or *co-ordinates* of each station from those two axes, and plotting the position of each station by the aid of a straight-edged scale fixed parallel to one of the axes, and a T square sliding along it, so as to rule lines parallel to the other axis, and at any given distance from it, and of any given length. When the direction of the true meridian has been ascertained, it is best to make one of the axes represent it; and in that case the co-ordinates parallel to one axis will be the distances of the stations to the north or south of the fixed point or "*origin*" O, and those parallel to the other axis will be their distances to the east or west of the same point; whence the phrase, "Northings, Southings, Eastings, and Westings." If the true meridian is unknown, any fixed direction will answer the purpose, and may be called "the Meridian" for the occasion, and one of its ends "the North." The calculations to be performed are the following:—In the figure, let O Y represent "the Meridian," Y being towards "the North." One of the stations in the survey is to be taken as the origin O. Let A be the next station, and O A its distance from O. If Y O A, as in the figure, is an acute angle, A is to the northward of O; if an obtuse angle, to the southward; if Y O A, as in the figure, lies to the right of

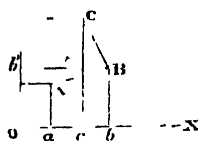


Fig. 38.

the meridian, A is to the eastward of O; if to the left, to the westward; and the co-ordinates of A are as follows (θ denoting the angle Y O A):—

$$\left.\begin{array}{l}\text{Northing } O\,a' = a\,A \text{ (or if negative, Southing)} = O\,A \cdot \cos\theta; \\ \text{Easting, } O\,a = a'\,A \text{ (or if negative, Westing)} \ldots = O\,A \cdot \sin\theta.\end{array}\right\} \quad (1.)$$

In the same manner are to be computed the co-ordinates of the third station B *relatively to* A; viz.:

$$a'\,b' = A\,B \cdot \cos\theta'; \quad a\,b = A\,B \cdot \sin\theta'; \quad \ldots\ldots\ldots\ldots \quad (2.)$$

(where θ' denotes the angle made by A B with the meridian); also the co-ordinates of C *relatively to* B, and so for each successive station. In the figure, it will be observed that the direction of B C deviates to the westward of north, so that $b\,c$ is a "Westing," and is to be considered as negative. The results of these calculations are to be entered in a book, in four columns—for northings, southings, eastings, and westings respectively. Then in four other columns are to be entered the total northing or southing, and easting or westing, of each station from the origin or first station, computed by adding all the successive northings and subtracting the southings, made in traversing to the station, the result being a northing if positive, a southing if negative; and by treating the eastings and westings in the same manner.

These calculations are expressed by symbols as follows.—Let $\pm y$ denote the total $\left\{\begin{array}{l}\text{northing}\\\text{southing}\end{array}\right\}$ of a station, and $\pm x$ its total $\left\{\begin{array}{l}\text{easting}\\\text{westing}\end{array}\right\}$; L the length of any given station line, and θ the angle which it makes with the meridian from the north; observing that both θ and $\sin\theta$ are $\left\{\begin{array}{l}\text{positive}\\\text{negative}\end{array}\right\}$ according as that angle lies to the $\left\{\begin{array}{l}\text{east}\\\text{west}\end{array}\right\}$ of the meridian, and that cosines of obtuse angles are negative. Then

$$\left.\begin{array}{l} y = \Sigma \cdot L \cos\theta; \\ x = \Sigma \cdot L \sin\theta. \end{array}\right\} \quad \ldots\ldots\ldots\ldots\ldots\ldots \quad (3.)$$

This method is chiefly useful in surveying mines, but may also be applied with advantage to some surveys above ground, such as those of towns. The book forms a record of the position of each station, independently of the plan; and it may be made more complete by the addition of a column containing the elevations of the stations above a datum horizontal surface. This will be further considered under the head of levelling.

46. The **Plane-Table** is a drawing-board, having a sheet of paper

strained on it, mounted on a portable three-legged stand, and capable of turning about a vertical axis, and of being adjusted by screws (like the azimuth circle of a theodolite) to a horizontal position, as shown by a spirit level laid on its surface.

The vertical axis has a clamp and a tangent-screw to adjust the table to any required position.

The *index* is a flat straight edged ruler, having upright sights at its ends.

The use of the plane-table resembles trigonometrical surveying on a small scale, except that the angles, instead of being read off on a horizontal circle and afterwards plotted, are at once laid down on paper in the field.*

Fig. 39 illustrates the principle of surveying with the plane-table. The first operation is to measure carefully a base on the ground, A B, and to lay down on the paper a straight line $a\ b$, to represent that base on a suitable scale. The instrument is then to be placed and levelled at the station A, the point a on the paper being directly above the point A on the ground; a needle is to be fixed upright at a; and the index being laid on the table, so that its fiducial edge shall lie exactly along the line $a\ b$, the table is to be turned until the sights of the index are in a line with the farther station B, and adjusted exactly to that position by the tangent-screw. The table remaining steady, the index is to be turned

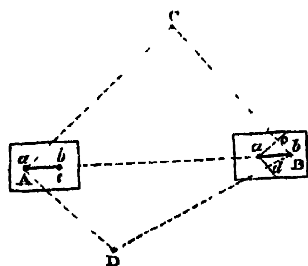


Fig. 39.

so that while its fiducial edge still touches the needle at a, its line of sight shall be successively directed towards all the important objects whose positions relatively to the base A B are to be found, such as C and D; and with a fine hard pencil lines are to be drawn along the edge of the index pointing towards those objects from a. The table is now to be shifted from A to B, and the needle from a to b, the point b on the paper being placed exactly over B on the ground; the index being laid along $b\ a$, the table is to be adjusted till the sights are in a line with A. The index is then to be turned so that while its fiducial edge still touches the needle at b, its line of sight shall be successively directed towards the same objects as before, and short lines pointing from b towards those objects are to be drawn along its edge, intersecting the lines

* To protect the paper against the effects of the alternate moisture and dryness of the air, Captain Siborn recommends that its lower side should have spread over it the beat-up white of an egg before it is laid on the board.

ously drawn; the points of intersection, such as *c* and *d*, mark the objects on the paper. The details are filled in by sketching.

The objects thus laid down include poles at points suited for additional stations. On removing the table to one of those new stations, such as C, the needle is to be fixed at the point *c* representing that station on the paper, and the index is to be placed with its edge touching that needle, and traversing also a point representing one of the former stations, such as *a*. The table is then to be turned so that the sights shall be directed towards a pole fixed at that former station; and then all the lines on the paper will be parallel to the corresponding lines on the ground; and the survey of additional objects from the new station may be proceeded with as before.

The plane-table is well suited for surveying where minute accuracy in details is not required, the end in view being to show the relative positions of the more important objects on the ground. It is therefore more useful for topographical and military purposes than for those of engineering. For full information as to its use see Pierce on "The Use of the Plane Table in Topographical Surveying," *Min. Proc. Inst. C.E.*, vol. xcii., 1888

CHAPTER IV.

OF LEVELLING.

47. **Setting-out a Line of Section.**—Preparatory to taking the levels of the ground along the line of a proposed vertical section, that line is to be "ranged," by marking on the ground with whites, poles, and permanent marks where required, the points where the line of section crosses all streams, lines of communication, boundaries, &c., and a sufficient number of other points to enable it to be exactly followed. For that purpose, a tracing is to be made of so much as may be necessary of the plan on which the intended line of section is drawn, and the distances of that line from corners of fences and other definite objects are to be carefully measured on the original plan, and marked in figures on the tracing. An assistant goes over the ground with this tracing, and marks the points in accordance with it. Should the leveller see fit to alter the line in any respect as he goes along it, or should it be left entirely to his own judgment to choose it, as is often the case with trial sections, the distances of a sufficient number of points in it from objects on the ground can be measured on the spot and noted on the tracing, so as to enable the line of section chosen to be laid down on the plan.

48. The **Spirit-Level** strictly speaking is a glass tube B C, fig. 40, hermetically sealed at both ends, containing some very limpid liquid, such as alcohol, chloroform, or sulphuret of carbon, and a bubble of air A, and having a slight curvature, convex upwards. That curvature is much exaggerated in fig. 40, being in reality so slight as to be imperceptible, or nearly so, to the eye. The air-bubble places itself at the highest point in the tube; and a tangent to the upper internal surface of the tube at that point is horizontal. The glass tube is usually fixed in and protected by a brass case. When the instrument to which the spirit-level belongs is in adjustment, the centre of the bubble is in the middle of the tube. When the bubble deviates from that position, it indicates that a tangent to the middle of the tube deviates from a horizontal position through an angle whose value in seconds is—

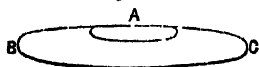


Fig 40

$$206264''\cdot 8 \times \frac{\text{deviation of bubble}}{\text{radius of curvature of tube}}; \qquad (1.)$$

so that the longer that radius, the more delicate is the spirit-level. A scale of equal parts is marked on or attached to the top of the tube, to measure the deviation of the bubble; and the value of those parts in seconds can be found by trial.

Fig. 41 shows a form of spirit-level introduced by Professor Piazzi Smyth, in which the air bubble A is very small. Another and a larger portion of air is contained in the upper part of the end C of the tube, which is separated from the rest by the partition D, with a nozzle-shaped orifice in its centre, through which air can be transferred so as to enlarge or diminish the bubble at will, by a mode of handling described in the *Transactions of the Royal Scottish Society of Arts* for 1856.



Fig 41.

The term *spirit-level* or *level*, is also applied to a *levelling instrument*, of which the spirit-level proper is the essential part. Various forms of level are used for engineering purposes; that which is represented in fig. 42 is Mr. Gravatt's, called the "Dumpy Level." A is the spirit level, attached by screws at *a, a* to the telescope B C; by means of those screws it can be adjusted, in order to place a tangent to its middle point parallel to the line of collimation of the telescope. A small circle near the object end B of the telescope, indicates a small transverse level, used to show whether the horizontal cross wire is truly horizontal.

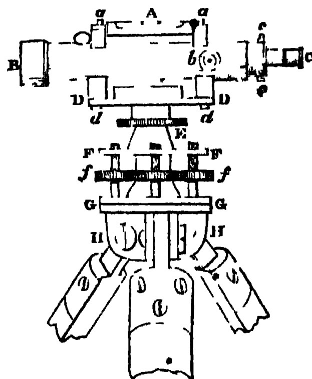


Fig. 42.

The telescope is similar to that of a theodolite (see Article 34, p. 53), except that the diaphragm at the common focus of the object glass and eye-piece contains one horizontal and two parallel vertical cross-wires, as shown in fig. 43. B is the object-end of the telescope, C the eye-piece; *b* the milled head of the pinion by which the inner tube is drawn in and out; *c, c* screws for adjusting the diaphragm so as to bring the horizontal cross-wire exactly to the line of collimation or axis of the telescope. D D is an oblong plate or flat bar fixed on the top of the vertical axis E; to this plate the telescope is

Fig 43.

connected by adjusting screws at *d*, *d*, by means of which the line of collimation is placed perpendicular to the vertical axis. The vertical axis is hollow, and turns upon a spindle fixed to the upper parallel plate F; that spindle is continued downwards and attached to the lower parallel plate G by a ball-and-socket joint. *f*, *f*, *f* are three of the four plate-screws by which the vertical axis is set truly vertical. The lower plate G is screwed on the staff-head H, which has three wooden legs like those of a theodolite.

In most levels a compass is carried on the top of the plate D D, for the purpose of taking the magnetic bearings of lines of trial sections.

The following are some of the principal variations from the construction above described:—

In Troughton's level, the brass case of the spirit-level is imbedded in the top of the outer telescope-tube, and has no adjusting screws; the adjustment of the spirit-level to parallelism with the axis of the telescope being left to the instrument maker.

In the Y-level, the telescope is carried by two forked supports called Y's. It can be rotated in these about its own axis, and can be lifted out and turned end for end. The spirit-level hangs below the telescope, instead of being supported above it. One of the Y's is supported by a vertical screw with a milled head, by means of which the telescope is adjusted so as to be at right angles to the vertical axis, and which answers the purpose of the screws at *d*, *d*, in the Dumpy level.

Instead of the four plate screws and ball-joint, many levels have three foot-screws, as in fig. 30, p. 58.

Some levels are provided with a small mirror, which being placed in a sloping position above the spirit level, enables the observer to see the reflected image of the bubble at the same time that he looks through the telescope. Reference has already been made to the contrivance of Professor Piazzi Smyth, by which an image of a small bubble is formed at the cross wires when the line of collimation is horizontal. (Article 37, p. 65; see also a paper by Mr. Bow, in the *Transactions of the Royal Scottish Society of Arts* for 1858-9.)

49. The **Levelling-Staff** is a rectangular wooden rod, having a face about two inches or two inches and a-half broad, on which is painted in a bold conspicuous manner, a scale of feet, divided into tenths and hundredths, commencing at the lower end of the staff. Its extreme length is usually from fifteen to seventeen feet, and it is made in three pieces, which in some staves can be put together or taken asunder, according as a greater or less length of staff is required, and in others, are made to draw out like telescope tubes. The staff, when in use, is held exactly vertical; for which purpose it sometimes has a plummet enclosed in a groove at one side of it, and

visible through a small piece of glass; it rests on its lower end, which is shod with brass; and in soft ground it is useful to have a small metal plate to place on the ground below the staff, and prevent it from sinking.*

When the telescope of the level is directed towards the staff, and the line of collimation is truly horizontal, the number of feet and decimals of a foot at which the horizontal cross-wire crosses the inverted image of the scale on the face of the staff (subject to corrections to be afterwards explained), shows the vertical depth of the point on which the lower end of the staff stands below the line of collimation. If two such observations are made with the staff at different points, and the level at the same station, the difference between the two readings shows, in feet and decimals of a foot, how much the point at which the *less* reading is taken is *higher* than the point where the *greater* reading is taken.

In an old form of levelling-staff, now seldom used, a "sliding-vane" was slid up and down by the staffman, in accordance with signals made by the leveller, until its centre was in the line of collimation prolonged; the staffman then read the height of the vane above the ground. The making the divisions on the staff so distinct that the leveller can read them himself is an invention of Mr. Gravatt.

50. The **Adjustments of the Level** may be distinguished like those of the theodolite, into *temporary adjustments*, which have to be made anew every time the level is set up, and *permanent adjustments*, which seldom become deranged in a well made level, but still ought to be tested on each day that the instrument is used.

I. The TEMPORARY ADJUSTMENTS are as follows:—

(1.) *To make the foci of the object-glass and eye-piece coincide with the cross-wires.*—The same as in the theodolite (see p. 59).

(2.) *To place the vertical axis truly vertical.*—The same as in the theodolite (see p. 58). In order to avoid straining the plate-screws, this adjustment ought first to be made as nearly as possible by shifting one of the legs of the stand, and then corrected by the plate-screws.

II. The PERMANENT ADJUSTMENTS are as follows:—

(1.) *To place the cross-wires in the axis of the telescope tube.*—In the Y-level, the same as in the Y-theodolite (see p. 60). In Troughton's level this adjustment is not made, except indirectly, as will be afterwards explained.

In the Dumpy Level, this adjustment may be made in the

* In order to ensure accurate reading of the staff, its points of division should always be at the *centre* of a black or white mark, and never at a *boundary* between black and white: for the apparent position of such a boundary always deviates from the real position in a direction towards black.

same manner as in the Y-level, by the maker of the instrument, before soldering the telescope-tube to the two blocks which support it upon the bar D D (fig. 42, p. 81); and, in that case, the adjusting-screws *c c* of the diaphragm should never afterwards be disturbed.

The same adjustment might be made by the observer, if there were any means of turning the inner telescope-tube about its longitudinal axis But the provision of such means would unnecessarily complicate the instrument; for it has been shown (by Professor Blood) that the exact coincidence of the cross-wires with the axis of the telescope-tube is not absolutely essential to accurate levelling.

This is demonstrated as follows:—

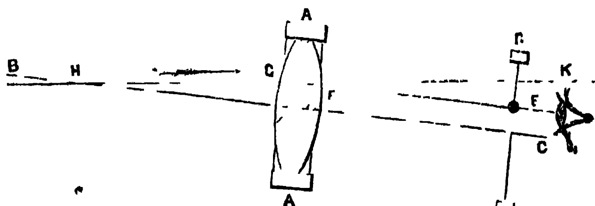


Fig 43 A.

In fig. 43 A, let A A represent the object-glass of a telescope, B C the axis of the telescope-tubes, and D D the diaphragm.

Suppose that the horizontal cross-wire E, instead of traversing the axis B C of the tubes is situated at a certain distance from that axis. Then, when the inner tube is drawn in and out, the cross-wire E will move along the straight line E F parallel to the axis C B.

Let H be the *outer principal focus* of the object-glass, situated in the axis C B. Then it is known that, by the laws of dioptrics, all rays of light whose paths within the telescope are parallel to B C, pass through the focus H, outside the telescope; so that, for example, a ray of light whose path within the telescope is F E, has for its path outside the telescope the straight line H G; and hence it follows that all possible positions of the cross-wire E, as the inner tube slides in and out, coincide with the images of points situated in *one straight line* G H. Consequently that line (or its prolongation within the telescope, G K) may be regarded as the *true line of collimation;* and if the spirit-level is adjusted so as to be parallel to that line, correct results will be obtained in levelling, although the cross-wire may not traverse the axis of the telescope.

(2.) *To make the line of collimation and the spirit-level parallel to each other.*—In the Y-level, bring the bubble to the middle of the spirit-level by means of the plate-screws; lift the telescope out of the Y's, and set it down with the ends reversed. If the bubble remains in the middle of the spirit-level, the adjustment is correct; if it deviates, correct one-half of the deviation by the plate-screws, and the remainder by the adjusting screws which connect the spirit-level with the telescope.

In Troughton's level, make two bench marks about ten chains apart; set up the level exactly midway between them, and read staves set upon them, so as to find, by the difference of those readings, the true difference of level of the bench marks. Now set up the level beyond one of the bench marks and read both staves; if the difference of the readings deviates from the true difference of level, alter the position of the diaphragm, by means of its adjusting screws, until the readings of the staves give the true difference of level. The cross-wires are thus placed in a line passing through the centre of the object glass parallel to the spirit-level; and the maker is relied on to make that line the true axis of the telescope. The same adjustment may be made by the aid of a sheet of water on a calm day; because two stakes can be driven at its margin so that their heads, being flush with the water, shall be exactly at the same level.

In the Dumpy level, having ascertained the true difference of level of two bench marks, as already described, and shifted the level to a position beyond one of them, alter, if necessary, the inclination of the telescope by means of the *plate-screws*, until the readings of the staves give the true difference of level, and bring the bubble to the middle of the spirit-level by means of the adjusting screws which connect the spirit level with the telescope (*a*, *a*, in fig. 42).

(3.) *To place the telescope and spirit-level perpendicular to the vertical axis* (or, as it is called, to make the instrument "traverse") place the telescope over a pair of plate-screws, and by turning them, bring the bubble to the centre of the spirit-level; reverse the direction of the telescope exactly, by turning it through 180° about the vertical axis, if the bubble is still in the middle of the spirit-level, the adjustment is correct: if not, correct half the deviation by the plate-screws, and the other half by means of the screws which connect the telescope with the bar on the top of the vertical axis (*d*, *d*, fig. 42).

51. The **Use of the Level** in finding the difference of elevation between two points has been described in the two preceding articles.

The observations, or readings of the staff, taken by means of the level, are called "*sights.*'

When two sights only are taken from one station, one with the staff upon a point whose level has been ascertained, and the other with the staff upon a point whose level is to be ascertained, the former is called the *back-sight*, and the latter, the *fore sight*.

If the back-sight is the greater, the ground rises, and if the fore-sight is the greater, it falls, from the former point to the latter.

When the levels of a series of points are taken with the level at one station, in order to make a continuous section, the first and the last observations are the principal back and fore sights respectively; the first back sight being taken with the staff on a bench mark or other point, whose level has been ascertained by means of a sight from a former station; and the last fore-sight being taken with the staff upon a mark which is to be the object of the first back-sight when the level is shifted to a new station. Of the intermediate sights, taken with the staff upon places where the inclination of the ground changes, on roads, at the bottoms of streams, &c, each is a fore-sight relatively to the preceding sight, and a back-sight relatively to the following one.

For example, in fig. 14, A is a station where the level is set up, and the horizontal line *b* A *c* is the *line of sight*, or straight line in prolongation of the line of collimation. The *first back sight* is taken with the staff on B, a point whose height above the datum-

Fig. 14.

surface has been ascertained by previous observations; and it gives, as the reading on the staff, B *b*. The *last fore-sight* is taken with the staff on C, a point well suited as a position for the staff when the *first back sight* with the level at the next station D is taken. It gives, as the reading of the staff, C *c*. The first intermediate sight, at the point marked 1, is a fore sight relatively to that at B, and a back-sight relatively to that at the point 2, and so on.

The first back-sight and last fore-sight are the most important in point of accuracy; for any error committed in them is carried on through the whole of the remainder of the section; whereas any error committed in taking an intermediate sight affects that sight only.

The first back-sight and last fore-sight taken from each station

ought to be at points as nearly as possible at equal distances from the level, in order to neutralize the effects of errors of adjustment, and also those of the curvature of the earth and of refraction, which will be explained in the next article; and those points should be on firm ground, and, if possible, so placed that the readings shall not exceed ten or eleven feet.

When the leveller thinks it desirable to carry his level on to a new station, such as D, the staffman holds his staff steadily at C, only making it face about; the leveller advances to D, sets up and adjusts his level, takes the first back sight C c, and proceeds as before. E e represents the position of the staff when the last fore-sight is taken from D; the staff is held there until the leveller has moved on and planted his level at a third station, and so on. These operations can be performed with one staff, but much time is saved by using two, carried by two staff-holders.

While the levels are thus being taken, two chainmen measure the line of section with the chain, in the manner described in Article 22, pp. 19, 20; except that instead of always chaining in straight lines, they follow the line of section as set out. The leveller notes the distances of all the points at which the staff is set up, as well as those where boundaries are crossed, whether levels are taken there or not. in this he may get useful help from the staff-holders.

In crossing a stream or a sheet of water, the leveller, besides taking enough of levels to give a section of its banks and bed, should take the existing level of the surface of the water, and also the highest and lowest levels of the water, so far as he can ascertain them Levels of the bottom may be taken by sounding.

When a sight is to be taken to determine the level of a point which is below the line of collimation by more than the entire length of the staff, the staff may be raised up vertically until the leveller can read some division near its upper end, and the height of the lower end of the staff above the ground may, at the same time, be measured with a tape-line or with another staff, and added to the height read. This, however, should only be practised at intermediate sights.

On the subject of **Checking Levels**, see Article 16, page 15. In good ordinary levelling the discrepancy between two sets of levels over the same section may be about a foot in forty miles of distance.

52. **Corrections for Curvature and Refraction.**—Inasmuch as a truly horizontal surface is not plane. but spheroidal (Article 3, p. 2), the line of sight of the level, when truly adjusted, does not exactly coincide with such a surface, but is a tangent to it. The

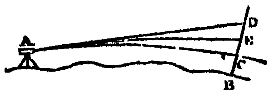


Fig. 45.

height read upon a levelling-staff, therefore, is always greater than it would be if a horizontal surface were plane; and the quantity to be deducted from the height on the staff of the point which is in the prolongation of the line of collimation, in order to reduce it to the height which would have been read had a horizontal surface been plane, is called the *correction for curvature.*

On the other hand, the line of sight, being the line along which light proceeds from the object looked at to the telescope, is not perfectly straight, being made slightly concave downwards by the refracting action of the air. Hence the point seen on the staff apparently in the line of collimation produced, is not exactly in that line, but is below it by an amount called the *error from refraction*, and thus the error arising from curvature is partly neutralized; and the correction to be subtracted for curvature and refraction usually is somewhat less than the correction for curvature alone.

In fig. 15, A represents the level; B, a point on the ground; B C E D, the staff standing on it; A C, a level surface touching the line of collimation, with the curvature very much exaggerated; A D, a straight line in prolongation of the line of collimation; E A, the real line of sight, a curved line in which light proceeds, owing to atmospheric refraction. Then the *correction for curvature* is $-$ C D; the *correction for refraction* $+$ D E; and the joint-correction,

$$- \mathrm{E\,C} = - \mathrm{C\,D} + \mathrm{D\,E}, \quad \ldots\ldots\ldots\ldots\ldots\ldots (1.)$$

The correction for curvature **is a** third proportional to the earth's diameter and the distance between the level and the staff—that is to say, its value *in feet* is

$$\frac{\text{distance}^2}{41{,}778{,}000} = \frac{2}{3}\,(\text{distance in statute miles})^2 \;\ldots\ldots\; (2.)$$

The error produced by refraction varies very much with the state of the atmosphere, having been found to range from one-half to one-tenth of the correction for curvature, and in some cases to vary even more. Its value cannot be expressed with certainty by any known formula; but when it becomes necessary to allow for it, it may be assumed to be on an average about one-sixth of the correction of a curvature; so that the *joint correction for curvature and refraction*, to be subtracted from the reading of the staff, is on an average,

$$\frac{5}{6} \times \frac{(\text{distance in feet})^2}{41{,}778{,}000} = \cdot 56\,(\text{distance in statute miles})^2 \ldots (3.)$$

The errors produced by curvature and refraction are neutralized

when back and fore-sights are taken to staves at equal or nearly equal distances from the level. At distances not exceeding ten chains, they are so small that they may be neglected.

The uncertainty of the correction for refraction makes it advisable to avoid, in exact levelling, all sights at distances exceeding about a quarter of a mile.

53. The **Level Field-Book** is kept in various forms, according to the practice of different engineers. In one of the most usual and convenient, each page is divided into seven columns, headed as follows:—

Rise.	Back-sight.	Fore-sight.	Fall.	Reduced Level.	Distance.	Description of Object.

The first entry made is in the column of reduced levels, being the elevation, in feet and decimals, of the bench mark on which the first back-sight is taken above a datum horizontal surface; and opposite this, in the column of description of objects, is the designation of that bench mark In the second and all the following lines the only entries usually made in the field are the back-sights and fore-sights, the distances, and the description of objects; so that, beginning at the right side of the page, we have in each line the description of a point or object (if any description is necessary), its distance from the commencement of the line of section, the fore-sight read upon the staff when held at that point, and the back-sight read upon the staff when held at the point immediately preceding. On each occasion when an intermediate sight is taken without shifting the level, it will be entered as a fore-sight opposite the point to which it is taken, and also as a back-sight in the following line. In *reducing* the levels, which ought to be done each evening for the levels taken during the day, the first process is to take the difference between the back-sight and fore-sight in each line, and enter it as a $\left\{\begin{matrix}\text{rise}\\\text{fall}\end{matrix}\right\}$ according as the $\left\{\begin{matrix}\text{back-sight}\\\text{fore-sight}\end{matrix}\right\}$ is the greater. The reduced levels are then computed in succession from the level of the first bench mark by the successive addition of the rises and subtraction of the falls.

The calculations in each page are checked by adding up the first four columns; when the difference between the total rise and total fall ought to be equal to the difference between the sum of the back-sights and the sum of the fore-sights, and also to the difference between the first and last reduced levels in the page, the first reduced level in the first page being that of the first bench mark; and the last reduced level in each page being also entered as the first in the following page.

It is sometimes useful to enter in the column of descriptions the

magnetic bearings of the lines levelled, and to illustrate it occasionally by sketch sections of the more intricate parts of the ground.

It is often necessary to reduce the levels in the field, especially in taking trial levels. In such cases the calculations should be carefully checked afterwards.

54. **Plotting a Section** is commenced by drawing with a very accurate straight-edge a straight "*datum-line*," to represent the datum horizontal surface from which heights are reckoned, and marking on that datum line a scale of distances. The vertical scale should be drawn on the paper *at right angles to the datum line*, in order that it may be parallel to the lines representing heights, and expand and contract along with them. This is of great importance in engraved and lithographed sections, in which the paper often expands or contracts differently in different directions. The plotting of the distances and heights entered in the field-book is performed like that of the distances and offsets in a chained survey. (Article 31, p. 32.) As to scales, see p. 7.

Explanations are usually written above the objects to which they relate, such as roads, railways, canals, rivers, &c.

The nature of the principal information which is required in writing on sections for engineering purposes has been stated in Article 14, pp. 14, 15.

55. **Levelling by the Theodolite** may be performed in three different ways.

I. *By placing the line of collimation horizontal, and using the theodolite like a level.*—This may be done when a proper levelling instrument is not at hand.

II. *By setting the line of collimation at a known angle of inclination, and taking sights in other respects as if with a level.*—This process may save time in taking the levels of steeply sloping ground. In fig. 46, A represents the theodolite, *b* A *c* the sloping line of sight, B *b*, C *c*, and the other vertical lines, heights read off on the staff. The most convenient way to reduce levels taken by this method is first to reduce them as if the line of sight were horizontal, and then, according as its inclination is upward or downward add to or subtract from each reduced height a *correction for declivity*, found by multiplying the distance of the point from the commencement of the sloping line of sight by the *sine* of the angle of inclination, if distances have been measured on the slope, or by its *tangent*, if they have been reduced to horizontal distances. (Article 23, p. 20).

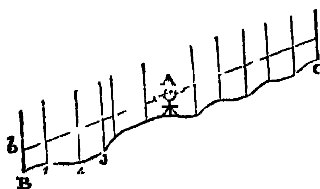


Fig. 46.

III. *By taking angles of altitude and depression.*—The height of an object above, or its depth below, the telescope of the theodolite, is nearly equal to its horizontal distance from the station of observation multiplied by the tangent of its altitude or depression, as the case may be.

The correction for curvature is *one-half of the angle subtended by the distance at the centre of curvature of the earth's surface*, or "contained arc," as it is called; and that correction is to be added to altitudes and subtracted from depressions. (As to the computation of that angle, see Article 33, Division V., pp. 46, 47.)

The correction for refraction is very variable, as has been already explained. On an average, it may be approximated to by diminishing the correction for curvature by one-sixth.

The effects of curvature and refraction may be nearly neutralized by taking *reciprocal angles*, as they are called; that is to say, if A and B be two stations, B being the higher; take the altitude of B as seen from A, and the depression of A as seen from B; half the difference of those angles will be the combined correction; and the *tangent of half their sum*, being multiplied by the distance, will give the difference of level nearly. The reciprocal angles should be taken as nearly as possible at the same instant, lest the refracting power of the air should change in the interval.

Levelling by angles is not to be relied upon for engineering purposes, except occasionally in taking flying levels.

The altitude of an object on land is taken with the sextant, by observing the "*double altitude*"—that is, the angle between the object and its image as reflected in a trough of mercury, called an "*artificial horizon*,"—and taking one half of that double altitude.

56. **Levelling by the Plane-Table** is performed by adjusting the table with particular care to a horizontal position, measuring the tangent of the altitude or depression, and multiplying it by the distance. To enable such tangents to be measured, the index is constructed as follows:—In fig. 47, E F is the flat bar of the index, F *b* its forward and E *a* its backward sight. Near the bottom of the backward sight is a sight-hole A for observing altitudes; near the top, a sight-hole *a* for observing depressions. A scale of equal parts is marked on the forward sight, and numbered upwards from B opposite A, and downwards from *b* opposite *a*. A slider D is slid up or down till a cross-wire contained in it appears in a line with the object, and the tangent is read by an index and vernier.

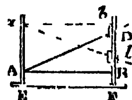


Fig. 47.

This process also is only suited for flying levels.

57. **Levelling by the Barometer and Thermometer** may occasion-

ally be used for engineering purposes to take flying levels in exploring the country. The following formula is sufficiently correct for that object:—

Let the quantities observed be denoted as follows:—

	At the lower station.	At the higher station.
Height of the mercurial column in the barometer,	H	h
Temperature of the mercury in degrees of Fahrenheit, as shown by the "attached" thermometer,	T	t
Temperature of the air in degrees of Fahrenheit, as shown by the "detached" thermometer,	T′	t'

Then the height of the higher station above the lower, in feet

$$= 60360 \left\{\log H - \log h - \cdot 000044\,(T - t)\right\} \cdot \left(1 + \frac{T' + t' - 64}{986}\right). \quad \ldots (1.)$$

For rapid calculation, the following, though less exact, is convenient:—

$$\text{Height in feet} = 56300\,(\log H - \log h) \cdot \left(1 + \frac{T' + t'}{900}\right) \text{nearly.} \quad (2.)$$

In the absence of logarithms, the following formula may be used for heights not exceeding about 3,000 feet. Correct the barometric reading at the higher station as follows:—

$$h' = h\left(1 + \frac{T - t}{10000}\right);\ \text{then}$$

$$\text{Height in feet} = 52428\,\frac{H - h'}{H + h'} \cdot \left(1 + \frac{T' + t' - 64}{986}\right) \text{nearly.} \quad (3.)$$

The preceding formulæ are applicable to the mercurial barometer. They are also applicable to the "Aneroid" barometer, with the exception of the correction depending on the temperature by the attached thermometer. The aneroid barometer, if very skilfully constructed, may be made to require no appreciable correction for the effect of its own temperature on its indications. Should it need such correction, the amount can only be determined by an experimental comparison between the individual aneroid barometer and a mercurial barometer. (See p. 789.)

Another method of taking flying levels, depending, like the barometric method, upon the pressure of the air, is that of determining the boiling-point of pure water by a very sensitive thermometer—a method invented by Dr. Wollaston, and improved by Principal Forbes. (See *Transactions of the Royal Society of Edinburgh*, vols. xv. and xxi.)

That boiling-point falls very nearly at the rate of *one degree of Fahrenheit for every 543 feet of ascent*; and still more nearly according to the following formula:—

$$z \text{ in feet} = 517\,(212^\circ - T) + (212^\circ - T)^2; \ldots\ldots (4.)$$

T being the boiling-point on Fahrenheit's scale, and z the height of the station where the experiment is made above a station where the boiling-point is 212°. To compare the levels of two stations, the boiling-point of pure water is to be observed at each, and the quantity z is to be calculated by formula 4 for each of the boiling-points; when the difference between those quantities z, corrected for the temperature of the air, will be the approximate difference of level.

58. **Detached Levels—Features of the Country.**—The use to the engineer of "Flying Levels," or observations of the heights of detached points, has already been mentioned in Article 10, p. 9. Such heights cannot be easily shown by means of vertical sections; and the most convenient method of recording them is to write them on a plan of the country.

Detached levels may be taken for the purpose of determining the elevations of important points on existing works, such as bridges, roads, railways, canals, &c., or of objects suitable for bench marks; or they may be taken in order to give the engineer a general knowledge of the form of the surface of the country. In the present Article it will be shown what positions are the best suited for detached levels taken with the last mentioned purpose.

On the surface of the earth, and, indeed, on any irregularly curved surface, two classes of lines may be distinguished, whose positions and figures are of primary importance in determining the shape of that surface—RIDGE-LINES and VALLEY-LINES.

I. A **Ridge-Line** is distinguished by the property, that along the whole of its course it is higher than the ground immediately adjacent to it on each side;—in other words, the ground slopes downwards from it at both sides. The rain-water which falls on the ground consequently runs away from both sides of a ridge-line; and hence it is also called a "WATER-SHED LINE." Ridge-lines are also sometimes called "the features of the country." The earth's surface is traversed by a number of main ridge-lines, which

are the central lines of the great mountain-chains; from these there diverge branch ridge-lines, and from these secondary branch ridge-lines, and so on; until in most cases the final ridge-lines end at promontories, where they sink down into the plains or the valleys. A ridge-line may return into itself, so as to contain within it an enclosed hollow or basin; but this is of comparatively rare occurrence.

A ridge-line is seldom either straight or level throughout any considerable part of its length, being almost always more or less wavy or serrated both vertically and horizontally.

The highest points of ridge-lines form the summits of the hills. The summit of a conical or rounded hill may in some cases be an isolated point, not traversed by a ridge-line; but the summit of a hill is in general traversed by at least one ridge-line, and is very often a point of divergence of several ridge-lines. A summit may sometimes be a flat expanse called a "table-land," with ridge-lines diverging from its edges.

II. A **Valley-Line** is distinguished by the property, that along the whole of its course it is lower than the ground immediately adjacent to it on each side;—in other words, the ground slopes upwards from it at both sides. The water on the surface of the ground consequently runs towards a valley-line from both sides, and except in certain cases, runs along the valley line in a stream; whence valley-lines may be called "WATER-COURSE LINES." The exceptions are, when the valley-line is in an enclosed basin, so that a lake is formed; and when the surface water disappears by evaporation or absorption. Between each adjacent pair of final ridge-lines there is a valley-line; these valley-lines converge and unite into greater valley lines, and so on until the final valley-lines end in the sea, or at the bottom of some enclosed basin, or at the edge of a plain. A valley-line, like a ridge-line, is seldom either straight or level throughout any considerable part of its length.

The end of a ridge line lies in general either in a plain, or between two converging valley lines, or in the bend of a valley-line. The commencement of a valley-line lies in general between two diverging ridge-lines, or in the bend of a ridge-line, or at a "Pass."

A **Pass** is a place on a ridge-line lower than any neighbouring point on the same ridge line, and might be described as a point where a ridge-line and a valley-line cross each other at right angles; but it is more in accordance with the ordinary use of the word "valley" to describe the line of lowest elevation at a pass, which crosses the ridge-line at right angles, as consisting of two valley-lines which run downwards from the pass in opposite directions.

From these descriptions of ridge-lines and valley-lines, and of

points in and connected with them, it is obvious that the places whose elevations are of most importance towards a knowledge of the figure of the surface of a district are the following:—

The summits of hills, being peaks, table-lands, or highest points of ridge-lines.

The points where the inclinations of ridge-lines change.

The points from which ridge-lines diverge.

The passes, or lowest points of ridge-lines and highest points of pairs of valley-lines.

The lowest points of valley-lines.

The points where the inclinations of valley-lines change.

The points where converging valley lines meet.

Of all those places, those which are of most importance in the engineering of lines of communication are the *passes;* because they are in general the points at which ridges are to be crossed.

To the levels of the places already enumerated may be added, those of the surfaces of seas, lakes, rivers, and other bodies of water, in their various conditions.

The valley-lines of a district are usually marked with sufficient distinctness on a plan by the water courses. The position of the ridge-lines is in general indicated by shading the slopes which fall from them in each direction, the best system of shading being that according to which the depth of the shadow varies with the steepness of the slope, being made as nearly as possible proportional to the tangent of the angle of declivity. The bottoms of valleys are, in addition, very slightly shaded, in order to distinguish them from the tops of hills.

59. **Contour-lines** are used as means of enabling a plan to give more complete information as to the figure of the surface of the ground than is possible by means of levels written in figures alone.

A contour-line on a plan represents a contour-line on the earth's surface, which is a line traversing all the points on the ground that are at a given constant height above the datum-level. A contour-line on the ground may be otherwise described as a horizontal section of the earth's surface, or the line where the earth's surface is cut by a given horizontal surface, or the outline of an imaginary sheet of water, covering the ground up to a certain given elevation.

All contour-lines cross the lines of steepest declivity on the surface of the ground at right angles; they also cross at right angles all ridge-lines and valley lines at which the surface of the ground is sensibly curved, and does not form an absolutely sharp ridge or furrow.

The vertical distance between successive contour-lines on a plan depends on the scale of the plan, the figure of the ground, and the

purpose for which the plan is intended; being greater in plans on a small scale than in those on a large scale; greater where the slopes are steep and the hills high than where the slopes are gentle and the hills low; and greater or less, also, according to the precision with which levels have been taken for finding the position of the contour-lines, and the use that is to be made of them in designing works For example, in the Ordnance Maps of Britain, on the scale of six inches to a mile, contour-lines are drawn at each twenty-five feet of height, and certain of these, called "principal contour-lines," are determined with greater precision than the others; and those principal contour lines are at every fifty feet of elevation in the flatter parts of the country, and at every hundred feet in the more hilly parts The st contour lines are those which have been laid down in s lans of town districts for purposes of drainage and other improvements. these occur at vertical intervals of from eight feet to two feet

Different methods of determining the positions of contour-lines may be followed according to the degree of precision required. To lay down principal contour-lines, a series of bench marks should be made at such points in ridge and valley-lines as have been already specified in the preceding Article; the positions of those bench marks should be ascertained in the course of the survey, and laid down on the plan, and their elevations found by levelling. Then by levelling from those bench marks, points are to be marked by or otherwise, on the ridge and valley-lines, and at as many places as may appear necessary, at certain definite ove the datum-level, such as 50 feet, 100 feet, 150 feet, and so on. The positions of the points so marked being surveyed with the chain and plotted, give a series of points in the contour-lines; and the course of those lines between the points so found by surveying is to be sketched upon a tracing of the plan taken to the ground for the purpose. Bench marks, whose levels ought to be checked, should be made at the places where principal contour-lines cross important ridge-lines and valley-lines.

Intermediate contour-lines can be interpolated between the principal contour-lines by sketching on the ground, aided by the known levels of the points where the rates of inclination of the ridge and valley-lines vary.

The horizontal distance between two adjoining contour-lines being inversely as the tangent of the angle of inclination of the ground, is also inversely as the depth of shadow to be used to express the steepness of the slope. "Hill-sketching," as it is called, consists in shading the slopes of hills upon the ground according to this principle, with the pencil, by drawing horizontal lines parallel to the contour-lines, and with a degree of closeness proportional to

that of the contour-lines themselves. Those pencil hatchings are in fact intermediate contour-lines sketched by hand.*

In engraved plans the shading of hills is effected by means of hatched lines at right angles to the contour-lines, and following, therefore, the lines of steepest declivity.

In order that an engineer may know how far he can depend upon the contour lines on a plan as a means of enabling him to select the best line for a proposed work, it is necessary that he should know by what method, and with what degree of precision, their positions have been determined, and that he should see upon the plan the positions and written levels of the bench marks and other detached points which have been used during that process.

60. **Cross Sections** may cross the centre line, or line of the longitudinal section, of a proposed work either at right angles or obliquely.

The term is applied to longitudinal sections of existing lines of communication which the proposed work has to cross. Such sections have already been referred to in Article 7, p. 6, and Article 8, p. 7.

Cross sections to assist the engineer in choosing the best line (referred to in Article 11, Division V., p. 10) should in general run along the ridge-lines and valley-lines which are to be crossed by the proposed work. They should also be made where the ground has a steep slope in a direction transverse or oblique to that of the centre line of the proposed work.

Cross sections to accompany the working section, for the purpose of enabling quantities of excavation or other work to be measured and calculated exactly (referred to in Article 11, Division XIV.,

* In the late Mr. Butler Williams's *Practical Geodesy*, p. 190, he describes in the following terms an approximate method of drawing contour-lines by the aid of pencil hill-shadings:—

"Horizontal contours can be traced by the eye with considerable accuracy, especially when the surveyor is assisted by the altitudes obtained in the trigonometrical operations serving for the construction of the outline map. The process . . . which I now proceed to describe, is rapid in execution, and tolerably correct for a small scale (say one inch or two inches to a mile), where experience has trained the eye to accuracy. It is well adapted for reconnoissances of a country, and is much used by military engineers. The civil engineer would, however, frequently find the same advantage in using it in his preliminary examinations of countries for the purpose of selecting general lines of communication.

"In the field, when the eye is alone depended upon, the horizontal lines are traced in pencil, by close parallel hatchings; and when the whole drawing is finished, the normal contours are traced at the required vertical distances apart, by following the general direction of the pencil lines, and checking their truth by means of the trigonometrical elevations or other heights marked on the map. The contours, when a complete circuit is made, must return to the point of departure; and if it were attempted by the eye alone to trace normal contours which are isolated from each other, no degree of previous experience would suffice for the attainment of the object.

p. 11), are in general at right angles to the line of the longitudinal section.

61. The **Water-Level** is an instrument used instead of the spirit-level where long range and great accuracy are unnecessary. It consists of an inverted siphon tube, fixed on the top of a stand, and nearly filled with water, which may be slightly tinged to make it the more easily visible. The horizontal part of the tube (about eighteen inches or two feet long) may be of metal: the two vertical branches (which are only two or three inches high) are of glass. The surfaces of the water stand at the same level in those two branches; and the leveller obtains a horizontal line of sight by looking along a line joining those two surfaces, which may be considered as the "line of collimation" of the instrument. When the distance of the staff is so great that the observer cannot read the divisions, a staff of the old kind, with a sliding vane, may be used. (See Article 49, p. 83.) The water-level is useful for setting-out the points of contour lines intermediate between the bench marks, being sufficiently accurate for that purpose, and more expeditious than the telescopic levelling instrument.

CHAPTER V.

OF SETTING-OUT.

62. **Ranging Straight Lines.**—It has already been stated in Article 11, Division XIII., p. 11, that the process of *ranging and setting-out the line* consists in marking on the ground the centre line of the proposed work.

That marking consists of two operations: temporary marking, or ranging, by means of poles; and permanent marking, or setting-out, properly so called, in which the principal marks are in general stakes.

The distance apart of the stakes used in setting-out the centre line of a proposed work varies considerably in the practice of different engineers. In some cases, a stake is driven at every chain of 66 feet; in others, at every 100 feet; while on some works the distance from stake to stake has been as great as 300 feet. Time and money are saved by adopting a long interval between the stakes, but at the expense of precision.

For ranging straight lines of moderate length, the most convenient instrument is a large sized transit theodolite—that is to say, one with circles of six inches in diameter or more (Article 34, pp. 54, 55)—because the telescope is capable of being turned completely over about its horizontal axis, so as to range one continuous straight line in two opposite directions from the station. In order that this operation may be correctly performed, great care must be bestowed on the adjustment of the line of collimation perpendicular to the horizontal axis (Article 35, p. 60), of the horizontal axis perpendicular to the vertical axis (Article 35, p. 61), and of the vertical axis truly vertical (Article 35, p. 58). With a good six-inch theodolite the error in ranging a pole in a straight line should not exceed 10″ in angular direction; that is to say, about three inches at a distance of a mile off.

For very long straight lines, however, the theodolite is not sufficiently exact; and then it becomes advisable to use a small Transit Instrument, consisting simply of a telescope with a horizontal axis, resting on a suitable stand, so as to be capable of being turned over in a vertical plane.

The telescope of a transit instrument for engineering purposes may be from twenty to thirty inches in the focal length of the

object-glass. At the middle of the length of the telescope tube is a hollow sphere, to which are joined two hollow cones, forming the arms of the horizontal axis. Those arms taper towards the ends, where they terminate in two hollow cylindrical pivots, which rest in angular bearings called Y's, each supported on the top of one of the standards of the frame. One of these Y's has a vertical adjusting screw, for raising or lowering it till the horizontal axis is truly horizontal; the other has horizontal adjusting screws, for shifting it back or forward until the horizontal axis is truly perpendicular to the vertical plane in which the line of collimation is intended to move. There is a moveable spirit level for placing the axis horizontal, whose use will presently be described.

At the common focus of the object glass and eye-piece are a set of cross wires carried by a diaphragm, which has adjusting screws to move it so as to place the line of collimation (marked by the intersection of the central pair of cross wires) exactly perpendicular to the horizontal axis. At night the cross-wires are rendered visible by light which enters from a lantern through one of the hollow pivots of the horizontal axis, and is reflected towards the cross wires by a small oblique mirror. The strong cast-iron stand of the instrument rests on and is screwed to a smooth level stone slab, forming the top of a massive stone or brick pedestal, built on a firm foundation. The building which shelters the instrument should be entirely disconnected from the pedestal, otherwise the vibrations produced in it by the wind will be communicated to the instrument

To facilitate the placing of the instrument exactly in a given alignment, the frame sometimes rests on a lower frame, like the slide rest of a lathe, along which it can be slid sideways into the required position by the action of a screw.

The adjustments of the transit instrument are as follows:—

(1.) *To place the line of collimation exactly perpendicular to the horizontal axis.*—Direct the cross-wires towards a well-defined point in a distinct object; lift the telescope with its axis out of the Y's, turn it over, so as to reverse the position of the axis end for end, and set it down again: if the cross-wires cover exactly the same object, the adjustment is correct; if not, correct one-half of the error by the horizontal adjusting screws of one of the Y's, and the other half by the adjusting screws of the diaphragm. Repeat the process till the adjustment is perfect.

(2.) *To place the horizontal axis truly horizontal.*—The spirit-level has two feet, which are to be placed striding across the telescope so as to rest on the two pivots of the horizontal axis respectively. Bring the bubble to the middle of the level by turning the vertical adjusting screw of one of the Y's; reverse the position

of the level end for end: if the bubble remains at the centre of the level, the adjustment is correct; if not, correct one-half of the deviation by the vertical adjusting screw of the axis, and the other half by the adjusting screw which regulates the height of one of the feet of the spirit-level. Repeat the operation till the adjustment is perfect; and be careful to remove the spirit level before moving the telescope.

(3.) *To place the plane of motion of the line of collimation exactly in the vertical plane traversing two distant stations at opposite sides of the instrument.*

The pedestal is to be built as nearly in the true alignment as is practicable by ordinary methods of ranging, and the upper surface of the flat stone which forms the top is to be carefully levelled.

The transit instrument having been set on the pedestal, and its line of collimation adjusted perpendicular to the horizontal axis, is to be moved by hand until the telescope, being turned alternately in opposite directions, points nearly towards the signals marking the distant ends of the line; observing, that when the line of collimation deviates to the *same* side of both signals (for example, in a line running north and south, to the east of both, or to the west of both) such deviation is to be corrected by shifting the stand sideways; and that when the line of collimation deviates to *opposite* sides of the signals (for example, to the east of one, and to the west of the other) such deviation is to be corrected by turning the stand as if about a vertical axis. This is the first approximation to adjustment. A second approximation is made in the same manner, after having levelled the horizontal axis; and then the places are marked for the screw sockets. The instrument having been removed, the holes for those sockets are to be cut, and the sockets fixed in them with lead.

The instrument is then replaced, and approximately adjusted as before, and the screws for fixing it to the pedestal are inserted, but not tightened. The instrument is finally adjusted by the aid of the horizontal adjusting screws of the horizontal axis, and the fixing screws are tightened.*

63. **Ranging and Setting-out Curves.**—The curved parts of railways require to be set out with great precision. The form almost universally adopted for them is that of circular arcs, though in a few instances other forms, such as that of the parabola, have been used. There are reasons for thinking that the best form, in a mechanical point of view, is that called the "elastic curve," which a spring of uniform transverse section takes when bent: (on this

* Ample details on this subject are given in Mr. Simms's work *On Practical Tunnelling.*

subject, see Article 434, page 651). The only methods which will be described here are three of those of setting out circular curves—the method by angles—the method by offsets—and the method by bisections of arcs.

METHOD I.—*Setting-out Circular Curves by Angles at the Circumference.*—This is the only method by which circular curves can be set out at once as quickly and as accurately as straight lines.* It depends on the well known principle, that the angle subtended by any arc of a circle at any point in the circumference of the same circle, is one-half of the angle subtended by the same arc at the centre of the circle. For example, in fig. 48, A B is an arc of a circle, C a point in the circumference of the same circle lying beyond the arc. The angle A C B is one-half of the angle subtended by A B at the centre of the circle. When the point at which the angle is measured lies upon the arc, as at E, it is the angle B E F = A E G, between the line drawn from one end of the arc and the prolongation of the line from the other end, that is equal to half the angle at the centre of the circle. When the point at which the angle is measured is one of the ends of the arc, as A, it is the angle D A B, between the tangent of the arc and its chord, that has the same property.

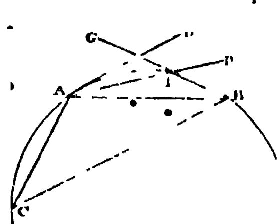


Fig. 48.

To express this by a formula, let a denote the length of the arc, r the radius of the circle, then

Angle at the circumference in minutes

$$= \text{A C B} = \text{F E B} = \text{D A B} = \frac{\text{Angle at the centre}}{2}$$

$$= 1718'{\cdot}873 \frac{a}{r} \quad \ldots\ldots\ldots\ldots\ldots\ldots (1.)$$

(The co-efficient is the value in minutes of one half of the arc equal to radius; see p. 37.)†

* This method of setting-out curves by angles was published for the first time in a paper read to the Institution of Civil Engineers on the 14th of March, 1843, by the author of this work, who had first practised it in 1841. Methods of setting-out curves by the theodolite had previously been employed by Captain Vetch and Mr. Gravatt, but they had not, so far as the author knows, been published before 1862.

† American engineers describe the sharpness of curves by stating the number of degrees in the angle subtended at the centre by an arc of 100 feet in length, which angle they call the "angle of deflection." Its value is

$$\text{Angle of deflection in degrees} = \frac{5729{\cdot}6}{\text{radius in feet}}.$$

In applying that principle to practice, the best instrument is a six-inch transit theodolite, which will range the positions of poles at the distance of half a mile to the accuracy of an inch and a-half. With a smaller instrument, the distances must be shorter, or the precision less.

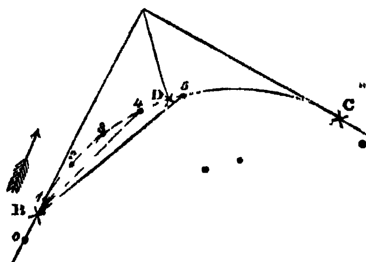


Fig 49.

PROBLEM FIRST.* To set out a circular curve touching two given straight lines, when the point of intersection of those straight lines is accessible.

In fig. 49, let B A, C A, be the two straight lines, intersecting in A. Set the theodolite at A, and measure the angle there, which denote by A; then lay off the two equal tangents, A B, A C, as calculated by the following formula (in which r is the intended radius of the curve) —

$$A\,B = A\,C = r \cdot \text{cotan} \frac{A}{2}; \;\ldots\ldots\ldots\ldots\ldots (2.)$$

and B and C will be the ends of the curve, where it touches the straight lines.

It is convenient (though not always necessary) to find the middle point of the curve. For that purpose, range, by means of the theodolite, the line A D bisecting the angle at A; and lay off the distance,—

$$A\,D = r \cdot \left(\text{cosec} \frac{A}{2} - 1\right); \;\ldots\ldots\ldots\ldots\ldots (3.)$$

then will D be the middle point of the curve.

The points B and C (and also D, if marked) should be marked by stakes distinguished in some way from the ordinary stakes which are driven all along the centre line at equal distances of one chain, or 100 feet, or some other distance.

A curve is called a "one-degree curve," a two-degree curve," and so on, according to its angle of deflection. Hence,

A "one-degree curve" means a curve of			5729·6 feet radius;	
a "two-degree curve"	—	—	2864·8 —	—
a "three-degree curve"	—	—	1909·9 —	—

and so on.

* For some additional problems in setting out curves, see Article 434, p. 653.

The total length of the curve is found by the formula,

Arc B C = ·0002909 r × supplement of A in minutes....(4.)

Any one of the points, B, C, or D, will answer as a station for the theodolite in ranging the curve. The commencement of the curve, B, is the station that involves the simplest operations, but when the length of the curve exceeds about half a mile, the middle point, D, is the best station as regards accuracy and convenience.

The following is the process of ranging the curve with the theodolite planted at its commencement, B:—

For brevity's sake, the distance between the stakes which mark the centre line of the proposed railway will be called "a Chain," whether it is 66 feet, 100 feet, or a greater distance.

Let o, in fig 49, represent the last stake in the portion of the straight line immediately preceding the curve; the distance B 1 from the commencement of the curve to the first stake in it will be the difference between one chain and o B. The angle at the circumference subtended by the arc B 1 having been calculated by equation 1, is to be laid off by the theodolite from the tangent B A, the zero point of the azimuth circle being directed towards A. The line of collimation will then point in the proper direction for the first stake in the curve, 1: and its proper distance from B being laid off by means of the chain, its position will be determined at once.

The angles at the circumference subtended by B 1 + 1 chain, B 1 + 2 chains, B 1 + 3 chains, &c., being also calculated, and laid off from the tangent B A in succession, will respectively give the proper directions for the ensuing stakes, 2, 3, 4, &c., which are at the same time to be placed successively at uniform distances of one chain by means of the chain.

The difference between an arc of one chain and its chord, on any curve which usually occurs on railways, is in general too small to cause any perceptible error in practice, even in a very long distance; but should curves occur of unusually short radii, it is easy to calculate the proper chord, and set it off from each stake to the next, instead of one chain, the length of the arc. For this purpose, the following approximate formula is useful. Let r be the radius, a the arc, and c the chord; then—

$$c = a\left(1 - \frac{a^2}{24\,r^2}\right) \text{ nearly. } \ldots\ldots\ldots\ldots\ldots(5.)$$

When the curve is ranged with the theodolite at D, or at any other intermediate point in the curve, or at its termination C, the process is precisely the same, except that the zero-point of the azimuth circle is to be turned towards B instead of A; and that

when the chain passes the theodolite station (for example, in going from stake 4 to stake 5 in fig. 49, with the theodolite at D), the telescope is to be turned completely over.

When the inequalities of the ground make it impossible to range the entire curve from the stations B, D, and C, any stake which has already been placed in a commanding position will answer as a station for the theodolite.

The stakes or poles, after having been ranged by the theodolite, should have their positions finally checked and adjusted by a modification of the method of offsets, which will afterwards be explained.

PROBLEM SECOND.—To set out a circular curve, touching two given straight lines, when the point of intersection of those lines is inaccessible.

In fig. 50, the lines to be chained on the ground are represented by full lines; those whose lengths are to be calculated only are dotted.

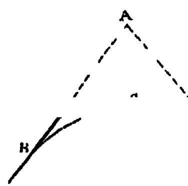


Fig. 50.

Let B A, C A, be the two straight lines, meeting at the inaccessible point A. Chain a straight line D E upon accessible ground, so as to connect those two tangents. The position of the *transversal* D E is arbitrary, but it is convenient so to place it that it will cut the proposed curve in two points, which may be determined, and used as theodolite stations.

Measure the angles B D E, D E C, which may be denoted by D and E. Then the angle at A is

$$A = D + E - 180°, \quad \ldots \ldots \ldots \ldots (6.)$$

$$A\,D = D\,E \cdot \frac{\sin E}{\sin A}; \quad A\,E = D\,E \cdot \frac{\sin D}{\sin A}. \qquad (7.)$$

$$D\,B = r \cdot \text{cotan}\,\frac{A}{2} - A\,D; \quad E\,C = r \cdot \text{cotan}\,\frac{A}{2} - A\,E, \ldots (8.)$$

and by laying off the distances D B and E C as thus calculated, the ends of the curve B and C are marked, and it can be ranged from either of those stations as in Problem First.

But it is often convenient to have intermediate points in the curve for theodolite stations; and of those the points of intersection with the transversal, H and K, and the point G, midway between these, can easily be found by the following calculations, in making which a table of squares is useful.

Let F be the point on the transversal midway between H and K.

If B D = C E, the point F is at the middle of D E. If B D and C E are unequal, let B D be the greater; then the position of F is given by either of the two following formulæ:—

$$\mathrm{D\,F} = \frac{\mathrm{D\,E}}{2} + \frac{\mathrm{B\,D}^2 - \mathrm{C\,E}^2}{2\,\mathrm{D\,E}}; \quad \mathrm{E\,F} = \frac{\mathrm{D\,E}}{2} - \frac{\mathrm{B\,D}^2 - \mathrm{C\,E}^2}{2\,\mathrm{D\,E}} \ldots (9.)$$

The points H and K are at equal distances on each side of F, given by either of the following expressions:—

$$\mathrm{F\,H} = \mathrm{F\,K} = \sqrt{\left\{\frac{\mathrm{D\,E}^2}{4} + \frac{(\mathrm{B\,D}^2 - \mathrm{C\,E}^2)^2}{4\,\mathrm{D\,E}^2} - \frac{\mathrm{B\,D}^2 + \mathrm{C\,E}^2}{2}\right\}}$$

$$= \sqrt{(\mathrm{D\,F}^2 - \mathrm{B\,D}^2)} = \sqrt{(\mathrm{E\,F}^2 - \mathrm{C\,E}^2)} \ldots\ldots\ldots (10.)$$

The equations 9 and 10 are deduced from the two following, which may be used in order to check the calculations, and are given in a form suitable for the use of a table of squares:—

$$\left.\begin{aligned} \mathrm{B\,D}^2 &= \mathrm{D\,H} \cdot \mathrm{D\,K} = \frac{(\mathrm{D\,H} + \mathrm{D\,K})^2 - (\mathrm{D\,K} - \mathrm{D\,H})^2}{4}; \\ \mathrm{E\,C}^2 &= \mathrm{E\,H} \cdot \mathrm{E\,K} = \frac{(\mathrm{E\,H} + \mathrm{E\,K})^2 - (\mathrm{E\,H} - \mathrm{E\,K})^2}{4}. \end{aligned}\right\} (11.)$$

The point G in the curve is found by setting off the ordinate F G perpendicular to D E, of the following length:—

$$\mathrm{F\,G} = r - \sqrt{r^2 - \mathrm{F\,H}^2} \ldots\ldots\ldots\ldots (12.)$$

The angles subtended at the *centre* of the curve by the several arcs between the commencement B and the points H, G, K, C, are as follows:—

$$\left.\begin{aligned} &\text{Angle subtended at the centre by BH} = 180^\circ - \mathrm{D} - \text{arc} \cdot \sin\frac{\mathrm{FH}}{r}; \\ &\text{— — — — BG} = 180^\circ - \mathrm{D}; \\ &\text{— — — — BK} = 180^\circ - \mathrm{D} + \text{arc} \cdot \sin\frac{\mathrm{FH}}{r}; \\ &\text{— — — — BC} = 180^\circ - \mathrm{A} = 360^\circ - \mathrm{D} - \mathrm{E}, \end{aligned}\right\} (13.)$$

and the length of any one of those arcs may be computed by means of the formula,

$$\text{Arc} = \cdot 0002909\, r \times \text{angle at centre in minutes.} \ldots (14.)$$

The use of such computations will appear in the next problem.

Cases may occur in which obstacles upon the ground render it necessary to make one or both of the ends of the transversal D′ E meet the straight tangents *beyond* the ends of the curve. The whole of the formulæ already given continue to be applicable, with only the following modifications.

When D lies further from A than B does, D B is negative in the first of the equations 8—that is, A D is greater than $r \cdot \text{cotan} \frac{A}{2}$; and the point H, as found by means of equation 10, lies, not on the arc to be ranged, but on the continuation of the same circle beyond B.

When E lies further from A than C does, E C is negative in the second of the equations 8—that is, A E is greater than $r \cdot \text{cotan} \frac{A}{2}$; and the point K, as found by means of equation 10, lies, not on the arc to be ranged, but on the continuation of the same circle beyond C.

The point G always lies on the arc to be ranged. The longer the ordinate F G is, the more carefully must it be set off at right angles to the transversal.

PROBLEM THIRD.—To set out circular curve touching two given straight lines, when part of the curve is inaccessible to the chain.

If the point of intersection of the tangents is accessible, the two ends of the curve are to be determined and marked as in Problem First, and also the middle point of the curve, unless it lies on the inaccessible ground; and the length of the curve is to be computed by equation 4.

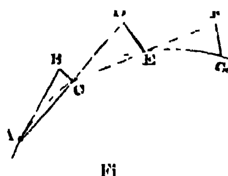
If the point of intersection of the tangents is inaccessible, the two ends of the curve, and at least one intermediate point, are to be determined and marked by the aid of a transversal, as in Problem Second, and the lengths of the arcs bounded by those points are to be computed by the formulæ 13 and 14.

A transversal may be useful even when the point of intersection of the tangents is accessible.

Each of the points thus marked will serve either as a theodolite station, or as a station to chain from, or for both purposes; and the stakes lying between the obstacle and the next station beyond it are to be planted by chaining backwards from that station.

Suppose, for example, that the commencement of the curve (B), lies at 243 chains 60 links from the commencement of the line, or "peg 0." The first stake in the curve will be 40 links from B, and will be "peg 244." Now, suppose that pegs 245 and 246 can be planted by chaining forwards, but that an obstacle occurs in the

course of the next chain. Let G denote a station in the curve beyond the obstacle, found by means of a transversal or otherwise, and let the arc B G, computed by the proper formula, be 6 chains 20 links. Then G is 213 · 60 + 6 · 20 = 219 chains 80 links from "peg 0," and lies between peg 249 and peg 250. Peg 219 is planted by chaining backwards 80 links from G, and pegs 248 and 247 by continuing to chain backwards. Peg 250 is planted by chaining forwards 20 links from G, and pegs 251, &c., by continuing to chain forwards. The ranging of the angular directions of the stakes from a theodolite station presents no peculiarity.



Fi

METHOD II.—*Setting-out Circular Curves by Offsets.*

In fig. 51, let A C, C E, E G, be a series of equal or unequal chords inscribed in a circle. Produce A C, to D, making C D = C E; join D E. The distance D E is called the "*offset;*" and its value is almost exactly

$$D E = \frac{C E \cdot A D}{2 r} \qquad (15.)$$

Let C E and E G be two *equal* chords; then the offset is

$$F G = \frac{C E^2}{r}. \qquad (16.)$$

If A B is a tangent to the curve at A, and C B a perpendicular let fall upon it from C, that perpendicular, being the *offset from the tangent*, is

$$B C = \frac{A C^2}{2 r}. \qquad (17.)$$

PROBLEM FOURTH.—To set out a circular curve by offsets, commencing at a given point on a straight line (fig. 51).

Let A be the commencement of the curve, found as in Problem First, and marked with a pole; A B the prolongation of the straight line (being a tangent to the curve), and B the end of the chain when laid along that prolongation from the last stake in the straight line. Plant a small pole at B, calculate the offset B C by equation 17, shift the end of the chain, and the pole along with it, sideways from B to C, keeping the chain tight, and leave the pole at C.

Drag the chain onward in the prolongation of A C; range a pole at D in a straight line with A and C, and at one chain's distance from C; shift the pole and the end of the chain through the offset D E, calculated by equation 15.

Drag the chain onward; range a pole at F in a straight line with C and E, and at one chain's distance from E; shift the pole and the end of the chain through the offset F G, calculated by equation 16; leave the pole at G, and so on.

If this process could be performed with absolute precision, the curve would terminate by exactly touching the further tangent at the point of contact found as in Problem First. But this never takes place at the first trial, except by accident; for any small inaccuracy in laying off the offset produces an error in the position of each stake, increasing nearly as the square of the distance from the commencement of the curve. If the final error is considerable, the curve must be ranged over again, until by successive trials the final error has been reduced to one not exceeding about ten links; then the positions of the stakes are to be finally adjusted by chaining round the curve once more, and shifting each stake sideways through a distance proportional to the square of its distance from the commencement of the curve.

Although this method is clumsy and tedious as a means of ranging curves, it is very useful for testing the uniformity of curvature of curves already ranged, and for rectifying the positions of individual stakes to the extent of an inch or two.

METHOD III. — PROBLEM FIFTH.—*To set out a circular curve by successive bisections of arcs.*

This is a method to be used only in the absence of angular instruments. It depends on the following relation between the versed sine of an angle B and that of its half,

$$\text{versin}\,\frac{B}{2} = 1 - \sqrt{1 - \frac{\text{versin}\,B}{2}}. \qquad (18)$$

To apply this principle, let B A, C A, in fig. 52, be the two tangents, and B and C the ends of the curve, so placed that A B and A C shall be equal, but leaving the radius to be found by calculation. Measure the chord B C.

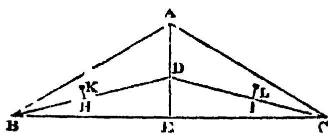


Fig. 52.

Then the simplest process for finding the radius is to use the following formula:—

$$r = \frac{A\,B \cdot B\,C}{2\sqrt{A\,B^2 - \frac{B\,C^2}{4}}}; \qquad (19.)$$

but as the triangle A B C is in general "ill-conditioned," it is

more accurate, though more laborious, to bisect B C in E, measure A E, and make

$$r = \frac{A\,B \cdot B\,E}{A\,E}. \qquad \text{(19 A.)}$$

Calculate the versed sine of the angle A B E = B, which is that subtended at the centre by one-half of the curve, as follows:—

$$\text{versin } B = \frac{A\,B - B\,E}{A\,B}; \qquad \text{(20.)}$$

and by means of equation 18 (using a table of squares, if one is at hand) calculate the versed sines of $\frac{B}{2}$, $\frac{B}{4}$, $\frac{B}{8}$, &c., in succession, observing that versin B enables one intermediate point in the curve to be found, versin $\frac{B}{2}$, three points, versin $\frac{B}{4}$, seven points; and generally, that versin $\frac{B}{2^n}$ enables $2^{n+1} - 1$ intermediate points in the curve to be found.

From the middle E of the chord B C and perpendicular to it, lay off the offset E D = r versin B; D will be the middle point of the curve.

Chain and bisect the chords B D, D C, and from their middle points and perpendicular to them, lay off the offsets

$$H\,K = I\,L = r \text{ versin } \frac{B}{2}; \qquad \text{(21.)}$$

K and L will be points in the curve, midway respectively between B and D, and between D and C; and so on until a sufficient number of points have been marked by poles.

Then chain round the curve as ranged by the poles, and drive stakes at equal distances apart.

The uniformity of the curvature may be finally checked by Method II.

64. **Nicking-out** the centre line of a proposed work, consists in cutting a small trench about six inches wide, to mark the centre line in the intervals between the stakes. The surface of the ground ought to be left undisturbed for a short distance on each side of each stake.

Where the centre line crosses fences and buildings, it should be distinctly marked by notches or grooves.

65. **Permanent Marks of the Line and Levels** are usually stakes

of larger dimensions than those which mark the centre line, and placed so far to either side of it that there is no risk of their being disturbed during the progress of the work, by which the marks on the centre line itself are obliterated.

The places where permanent marks of the course of the line are chiefly required are on the tangents of curves,—their distances from the ends of the curves being noted, so as to enable both the curves and the straight lines which connect them to be ranged over again at any time. Any important point on a curved or straight part of the centre line may be permanently marked by driving two stakes in a straight line passing through it, one at each side of the site of the work, and noting its distances from them, or by any of the means described in the second paragraph of Article 21, p. 17.

Stakes to be used as permanent bench marks for the levels of the work are about three or four feet long, and four inches square, hooped round the top with iron to prevent the head from being crushed. One of the best ways to form a firm surface for the staff to rest on is to drive into the head of the stake a long iron spike with a large convex head; the uppermost point of the convex surface of that head is the bench mark. Such marks are to be placed near the sites of all proposed pieces of masonry, and other structures of importance, and near the ends of cuttings and embankments; and opposite points where the rate of inclination or *gradient* of a proposed railway is to change.

As soon as any piece of masonry has been built high enough, one or more bench marks should be made on the masonry itself, to regulate the levels to which the remainder of the structure is to be built.

66. **Working Section and Level-Book.**—The nature of a working section has already been explained generally in Article 11, Division XIV., p. 11, and in Article 16, p. 15. The levels taken, in order to prepare it, consist for the most part of those of the stakes planted to mark the centre line, which are driven until their heads are flush with the ground. Should any inequality of the ground occur between two stakes, enough of additional levels and distances must be taken to enable an exact vertical section of it to be plotted; and the levels of every line of communication and other important object must be taken where it is crossed by the centre line. The level of every stake, and of every line of communication crossed, is to be taken twice over.

As to scales for working sections, see p. 7.

Cross Sections have already been referred to in Article 60, p. 97. When the ground is uneven sideways, they may be required at each stake. In general, they should be ranged accurately at right angles to the centre line, and should be plotted without exaggera-

tion; their vertical and horizontal scales being the same with the vertical scale of the longitudinal section. All cross sections should be plotted as seen by looking *forwards* towards them along the centre line.

The **Level-Book** of the working section of a line of communication is a book containing a complete statement of the levels of the ground and of the intended work, and of other information, which will presently be specified. Each folio of the book is divided into several columns, whose number, arrangement, and contents differ in the practice of different engineers. The following statement of the contents of the several columns of a level book may be taken as an example. It is specially adapted to a railway, but may be made, by slight modifications, to suit other kinds of works: —

Column 1. Numbers of the stakes planted at equal intervals of 66 feet, 100 feet, 300 feet, or some other distance along the centre line. The stake at which the line commences is numbered 0.

Column 2. Distances from the commencement of the section, in links or feet, as the case may be.

Column 3. Descriptions of objects between the equidistant stakes, such as fences, streams, roads, canals, railways, intermediate stakes at ends of curves, &c.

Column 4. Levels of the ground at the stakes, and between them when necessary, and of bench marks.

Column 5. Intended level of the upper surface of the railway (or other proposed work.)

Column 6. *Formation level* (that is, level of the ground when prepared by excavation or embankment for the completion of the work).

Column 7. Depths of cutting, }
Column 8. Heights of embankment, } as calculated by taking the differences between the numbers in column 4 and column 6.

Column 9. }
Column 10. } Rate of lateral slope of the ground, if any, to the { left / right } of the centre line, specifying whether it rises or falls from the centre line. If the slope of the ground is irregular, reference may be made to a cross section.

Column 11. }
Column 12. } Breadths of land required for works only (exclusive of fences) to the { left / right } of the centre line. These are called "*half-breadths.*" The method of calculating them will be described under the head of EARTHWORK, in the sequel.

Column 13.} *Total half-breadths* to the {left} of the centre
Column 14.} {right}
line, found by adding the intended breadth of the fencing to the half-breadths in columns 11 and 12.

Column 15. Angles at which streams, roads, canals, railways, &c., cross the centre line, stating (if the angles are oblique) whether the acute angle lies to the *left* or *right* of the centre line looking forwards.

Column 16. Remarks:—Comprising positions of permanent marks, rates of inclination or gradients, radii of curves, spans and head-room of bridges, tunnels, and arches of viaducts, alterations of level of existing lines of communication, &c.,—the whole accompanied by a sketch of the working section.

When an existing line of communication is to be altered in position or level for the purposes of the proposed work, a working section of the works required for such altered line should be prepared in the same manner with that of the principal work, and its description inserted in the level-book.

67. **Setting-out Slopes and Breadths of Land** (already referred to in Article 11, p. 11) is performed by laying off the *half-breadths* of the work and the *total half-breadths*, as calculated, exactly at right angles to the centre line, marking their ends with stakes and sometimes also nicking out lines so as to connect those stakes, and show the boundaries of the earthwork and the boundaries of the land to be occupied respectively. A temporary fence is made, as soon as possible, along the outer of those boundaries.

The **Land-Plans** (referred to in the same page) are prepared by plotting the total half-breadths on the plan of the working survey, drawing the boundaries of the pieces of land required for the work, and making separate copies or tracings of them, to be used in dealing with the owners and occupiers.

68. **Permanent Marks of Sites of Works** are stakes planted on nearly the same principle with those already described in Article 65 for marking points on the centre line. For example, suppose that the work to be set-out is a bridge, consisting principally of two abutments which support an arch or a platform. The principal points, upon which the positions of all other points in the bridge depend, are the four corners of its abutments. To enable the positions of those corners to be found at any time, plant four stakes in the prolongations of the faces of the two abutments, at known distances from the four corners, and sufficiently far from them to be clear of the work.

69. In **Setting-out Levels of Excavations** the engineer causes stakes to be driven, whose heads are at the intended formation-level.

To plant a stake at a given level, the staff is to be held upon the nearest bench mark, and read; the difference between the level of that bench mark and that of the new stake to be driven, is to be added to the reading of the staff, if that stake is to be lower than the bench mark,—subtracted, if it is to be higher. This gives the height which will be read upon the staff at the new stake, when that stake has been driven to the proper depth.

Two such stakes, being driven at fifty feet apart or thereabouts, in the centre line, near the commencement of a proposed cutting, enable the excavators to carry on the cutting at the proper level and rate of inclination for some distance, by the operation called "*boning*," which consists in ranging a line of uniform inclination from two given points in it, with T-shaped instruments called "boning rods." Each of these consists of an upright staff, having a cross-bar at right angles to it at the top: all the boning-rods belonging to one set ought to be exactly of the same height. To range or "bone" the bottom of a cutting with them from two given stakes, two of the rods are to be held upright on the heads of the two stakes, and a third held upright at any point in the cutting which is in the same straight line with the stakes; when, if the bottom of the cutting is at the true formation level, the tops of the three rods will be in one straight line. In this manner the cutting is carried forward at an uniform rate of inclination, until the engineer thinks it advisable to plant a new pair of stakes by the level and staff near its inner end, from which the boning goes on as before.

70. **Ranging and Setting-out Tunnels.**—The centre line of a tunnel having been at first ranged on the surface of the ground, in the manner already described, a row of shafts are sunk in convenient positions along that line.

In order to range the line below ground, it is necessary to have two marks in the centre line at the bottom of each shaft, as far asunder as possible, to enable that line to be prolonged from the bottom of the shaft in both directions. Those marks consist of nails or spikes driven into the cross-timbers.

The former practice was to determine the positions of those marks below ground, by erecting over the shaft a timber frame, from which two plumb lines were suspended, hanging nearly to the bottom of the shaft, and to range those plumb lines by the transit instrument; but as that process is difficult or impossible in windy weather, Mr. Simms introduced the following improved method:—* The engineer ranges, by the transit instrument, two strong stakes in the centre line above ground, each about sixteen feet from the centre of the shaft, so as to be safe from disturbance while the

* Simms *On Practical Tunnelling.*

work is in progress. To mark the exact position of the centre line, each stake has driven into its head a spike, with an eye through its top. The eye of each spike is very carefully ranged in the exact centre line, being made visible to the observer at the instrument by holding a piece of white paper behind it. A cord is stretched through the holes in the spikes, so as to mark the course of the centre line across the mouth of the shaft. At each side of the shaft a plank is laid, at right angles to the string, and with its edge overhanging the edge of the shaft two or three inches, so that a plumb-line may hang from it clear of the side of the shaft. Two plumb-lines are then hung from the planks, directly under the cord that marks the centre line; and the lower ends of those plumb-lines show two points in the centre line at the bottom of the shaft.

The approximate ranging of the "heading" or "drift," or small horizontal mine that connects the lower ends of the shafts, is performed by means of candles, each hung from the timber framing in a sort of stirrup.

The accurate ranging of the centre line, after the heading has been made, is performed by stretching a cord between the marks already ranged at the bottom of the shaft, and fixing, at intervals of thirty or forty feet, either small perforated blocks of wood carried by cross-bars, or stakes with eyed spikes driven into their heads, so that the holes in the blocks or spikes shall be ranged by the cord exactly in the centre line. The centre line of any part of the tunnel can then be marked at any time when required, by stretching a cord through two of those holes. The cross-bars are fixed in a temporary way to the timber framework of the heading, so that they can be removed, to leave a free passage for men and wagons; but their places are so marked that they can be re-fixed exactly in their proper positions at any time when it is required to range part of the line.

Curves can be set-out below ground by means of a theodolite on a short-legged stand, and candles or lamps instead of ranging-poles. In this case, the two marks at the bottom of a shaft indicate the direction of a tangent to the curve at its centre.

When the line of shafts does not follow the centre line of the tunnel, but a line parallel to it, a corresponding line is to be set-out through the heading at the bottom of the shafts; and from that line the centre line, or any given part of the tunnel, can be set-out by laying down offsets in transverse headings.

In order to *set-out the levels of a tunnel*, there should be a bench mark above ground, as described in Article 65, p. 110, near the mouth of each shaft. When the shaft has been sunk, and lined with timber or brickwork, a second bench mark is to be made within the shaft, and near its top, by driving into the timber

brickwork a horseshoe-shaped staple in a horizontal position, the levelling-staff being held on its upper surface in taking its level.

Some part of the masonry or brickwork of the intended tunnel is taken as a standard point by means of which the levels of other points are regulated: for example, the "invert-skew-back," or joint where the inverted arch forming the bottom of the tunnel meets the sides. That joint being at a fixed height above or below the rails (generally below), its depth below the staple is to be calculated. That depth is then to be set-off by hanging through the staple a chain of rods of the proper length. The rods used by Mr. Simms are connected together at the ends by eyes and spring-hooks: the length of each rod, from the inside of the eye at one end to the inside of the hook at the other, is ten feet. To set-off a given depth below the staple, the number of rods to be linked together is one more than the number of entire tens of feet in the depth; the odd feet and decimals of feet are set-off on the uppermost rod by screwing a gland upon it at the proper point. The chain of rods is then dropped through the staple until the gland, resting on the staple, prevents them from passing further, and supports the whole chain; a bench mark, consisting of a flat-sided spike driven horizontally into the timbering, or of a stake with a round-topped spike in its head, driven vertically into the ground, is then adjusted at the bottom of the shaft, so that its upper surface is exactly on a level with the bottom of the lowest rod.

The staple forms a permanent bench mark, through which the rods can be lowered again, whenever it is necessary to make a new bench mark under ground, owing to disturbance of the former bench mark. This is always done after the brickwork has been partly built, in order to make a permanent bench mark, by driving a flat spike into the side of the tunnel.

Further remarks on setting-out will be made under the head of each kind of work for which peculiar methods are required.

CHAPTER VI.

OF MARINE SURVEYING FOR ENGINEERING PURPOSES.

71. **Limitation of the Subject—Landmarks—Buoys.**—Marine surveys are undertaken for purposes of geography and navigation as well as for those of engineering; but the present chapter has reference to the last of those purposes only; and it therefore describes the operations of marine surveying so far only as they are required in preparing plans for engineering works in navigable waters.

The principal objects of such surveying are to determine and represent on a plan the figure of the bottom of the sea, or other piece of water, on a scale suited for designing engineering works, and to ascertain the materials of which the bottom consists, the level, rise and fall of the surface of the water, and the direction and speed of its currents.

The marine survey must be based upon a survey on the adjoining land, by means of which the figure of the coast and the positions of a sufficient number of conspicuous and well-defined objects near the coast have been ascertained. These objects are the *landmarks*, by observations of which the positions of points on the surface of the water are determined.

Stations afloat can be marked by means of buoys, carrying poles and vanes.

To prevent a buoy from deviating to any considerable distance from a position directly above its anchor, the mooring cable, which is fixed at one end to the anchor, passes through a ring called a "thimble," attached to the buoy, and has a weight hung to the other end.

72. **Datum and Bench Marks for Levels.**—There should also be a *datum-point*, or principal bench mark, on land, to which the levels are referred, and a sufficient number of other bench marks, whose elevations relatively to the principal bench marks are to be found by the ordinary process of levelling.

For nautical purposes the *datum-surface*, relatively to which the levels of the bottom are stated, is the *average low-water-mark of spring tides;* and the same datum surface, when it is sensibly horizontal, will answer for an engineering survey; but on the sea-coast, when the survey is extensive, and in the channels of rivers, the low-water of spring tides is not a horizontal surface; and in such

cases, the levels for engineering purposes must be reckoned from an arbitrary horizontal surface, as in sections on land.

73. **Tide-Gauges.**—The successive levels of the surface of the water must be observed and recorded from time to time, as well for their own importance as because the levels of the bottom are ascertained by sounding, and in order to reduce the latter levels to a common datum the variations of the level of the surface of the water must be known.

The tide-gauges used for this purpose, when of the simplest kind, are posts set exactly upright, and having scales of feet and tenths of feet marked upon them, numbered from the bottom upwards. They must be fixed and stayed in such a manner as to be capable of resisting the waves. Sometimes the whole rise and fall of the tide at a given place may be observed on one post; but in general the slope of the beach makes it necessary to have a row of posts extending from low-water mark to high water-mark, and forming, in fact, one tide-gauge, divided into several stages or steps. The lowest mark on the lowest post of the row is the zero of the tide-gauge: its level should be ascertained relatively to the nearest bench mark on land by levelling It will form the commencement of a scale of feet and tenths, numbered upwards. The lowest mark on the second post must be made at a point adjusted by levelling to the same level with the highest mark on the first post, and marked with the same number, and so on; so that the marks on the entire row of posts may form one continuous scale of heights above the zero-mark

The number of different tide gauges required, and the places where they are to be erected, will be fixed by the engineer to the best of his judgment, so as to give the means of determining the figure of the surface of the water at any given instant. They must be more numerous, the more the surface of the water at each instant deviates from a horizontal form. Such deviation always exists in river channels; and in them, and also in estuaries, and on the coast, its existence and extent are indicated by differences in the time of high and low-water, and in the extent of rise and fall of the tide. Even when those deviations are not practically appreciable, it is desirable to have two tide-gauges at points distant from each other, in order that the two series of observations may check each other.

The observers of the tide-gauges should be trustworthy and intelligent persons, provided with watches, which should be compared every day with that of the principal surveyor.

For the purpose of reducing soundings only, it is in general sufficient to observe each tide-gauge at each quarter of an hour. When it is desired also to ascertain the laws of the tide at the

locality, it is better to observe the height on the tide-gauge at each ten minutes, for an hour before and an hour after high and low-water, and at each half-hour during the remainder of the day.

For engineering purposes, the tide-gauges already described, consisting of simple posts, are in general sufficient; because when the water is smooth enough to take accurate soundings, it is smooth enough to enable the observer of the tide-gauge to estimate the mean level between the crests and troughs of the waves.

When more exact observations are required, the tide-gauge should consist of an upright tube, communicating with the water outside through a few small holes only, and having in it a float with a graduated upright stem, tall enough to be visible above the top of the tube.

In a self-registering tide-gauge, such a float acts through a chain or cord on a train of mechanism, and moves a pencil up or down, which marks a line on a paper-covered cylinder turned by clock-work. (Airy *On Tides and Waves.*)

The observations at the tide-gauges having been copied from the observers' books into one book, are to be reduced to the datum of the survey by the aid of the known levels of the zero-marks of the tide-gauges relatively to that datum.

The mean of all the reduced observations of the tide-gauges taken during one or more entire "lunations," or revolutions of the moon, gives the *mean level of the sea*, which is a truly horizontal surface.

Further remarks on the tides will be made in the sequel.

74. **Determining Stations afloat.**—In fig. 53, let D represent the position at a given instant of a point in a boat, which is to be determined.

This is done by measuring with the sextant in the boat the angles between three known objects on land, A, B, C.

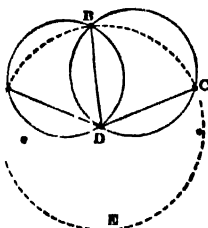


Fig. 53.

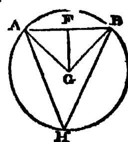


Fig 54.

To diminish or prevent the errors that would arise from the boat's shifting its position while the angles are being measured, the surveyor should have three sextants, if possible, with which he should take the angles A D B, B D C, A D C, in rapid succession, reading them off at leisure afterwards. The angle A D C, which should be the sum of the other two, serves as a check upon their accuracy.

Care should be taken that the four points, A, B, C, D, do not lie in or near the circumference of one circle; for in that case the

observations would leave the position of D indeterminate, as will presently be explained.

There are different methods of plotting the position of D on the plan.

METHOD I.—*By Two intersecting Circles.*—To draw through two points, as A and B, fig. 54, a circle which shall *contain* a given angle; that is to say, a circle such that from any point in its circumference, as H, the arc A B shall subtend an angle A H B equal to the given angle, draw through A and B the straight lines A G, B G, making with the straight line A B the angles B A G, A B G, each equal to the complement of the given angle; the intersection of those lines G will be the centre of the circle required.

Let fig. 53 now represent the plan, and A, B, C, the positions of the three landmarks as plotted on it; through A and B draw a circle containing the observed angle A D B; through B and C draw a circle containing the observed angle B D C; those circles will give by their intersection the point D on the plan:—unless they should happen to coincide with each other and with the dotted circle A B C E, when the point D may be anywhere in that dotted circle, and cannot be plotted from the observations taken. Should D lie *near* the dotted circle, the two intersecting circles will cut each other at too acute an angle, like the sides of an ill-conditioned triangle; and the plotted position of D will be liable to inaccuracy.

METHOD II.—*By the intersection of a Circle and a Straight Line.*—From A draw A E, making the angle C A E = C D B: from C draw C E, making the angle A C E = A D B, and cutting A E in E: through the three points A, C, E, describe a circle: through E and B draw a straight line cutting the circle in D; D will be the required point on the plan.

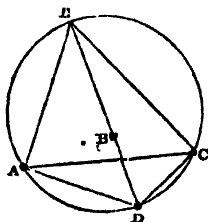


Fig. 55.

The two preceding methods are both too tedious for ordinary use, and the two following are almost always employed instead.

METHOD III.—*By a Piece of Tracing paper.*—On a piece of tracing paper draw three straight lines radiating from one point so as to make with each other angles equal to A D B and B D C. Lay it on the plan, and shift it about till the three lines traverse A, B, and C, respectively; the point from which they diverge being pricked through on the plan, will give the position of D.

METHOD IV.—*By the Station-pointer.*—This is an instrument consisting of three long flat arms turning about one centre, and having straight fiducial edges diverging from that centre. Fixed to one arm is a graduated circular arc, and fixed to the side

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arms, two indexes with verniers, by means of which those arms can be set so as to make any required pair of angles with the middle arm. The arms being set so as to form the angles A D B, B D C, the instrument is laid on the plan and shifted about until the three fiducial edges traverse the points representing the three landmarks respectively. The centre of the instrument will then be over the required point D, which is marked by means of a pricker that passes through a hole in the centre of the instrument; or otherwise, three pencil lines may be drawn along the fiducial edges of the arms, and produced after the instrument has been lifted off the paper; where their intersection will give the required point.

Three landmarks are all that are absolutely necessary to determine the position of a station afloat; but when the station is an important one, the surveyor, for the purpose of verification, should measure angles to additional known objects.*

Where a sufficient number of objects on land are not visible, the positions of stations afloat may be determined by taking angles to previously determined stations afloat which are marked by buoys, or at which boats with flags are moored; but this method is wanting in precision, and objects on land are always to be preferred when they can be seen.

75. **Soundings and Levels.**—The instrument generally employed for taking soundings for nautical purposes is the *lead-line*, a tough, hard, and flexible cord, loaded with a conical lead weight, and divided into fathoms. For engineering purposes, where the depth does not exceed about 100 feet, a chain is used. In shallow water, the best instrument is a rod, divided into feet and tenths, and loaded at the lower end.

The sounding lead is "armed" with a lump of tallow in a hollow at its lower end, by which, when the material of the bottom is loose, specimens of it are brought up. When the material is of a firmer texture, a specimen may be brought up by dropping a heavy iron pike, jagged and barbed at the lower end, called a "plunger," and hauling it up again by a rope; or the nature of the bottom

* The distances of the station from two of the landmarks might be calculated by the rules of plane trigonometry and plotted; but the process is too tedious for ordinary use. The following are the steps of which it consists (see fig 55):—

In the triangle A E C, given A C, and the angles E A C (= B D C) and A C E (= A D B), calculate A E and C E.

In the triangle A B E, given A B, A E, and ∠ B A E (= B D C — B A C), calculate ∠ A E B

In the triangle B E C, given B C, C E, and ∠ B C E (= A D B — B C A), calculate ∠ B E C.

In the triangle A D E, given A E and the angles, calculate A D.

In the triangle D E C, given C E and the angles, calculate C D.

may be ascertained by boring, or by diving—operations which will be again referred to further on.

Soundings for nautical purposes are noted, and written on the plan, in fathoms of six feet, and half and quarter fathoms; those for engineering purposes, in feet and decimals, or feet and inches.

The levels of the bottom are ascertained by taking several series of soundings along straight lines, in such positions as the engineer judges to be best. In general, the position of those lines is nearly that of the lines of steepest declivity of the bottom, and nearly at right angles to the coast.

As each sounding is taken, the surveyor notes the time, the depth, and the position.

The following are two methods of determining the positions of soundings:—

METHOD I.—*By a Series of Angles.*—In fig. 56, A and B represent two known objects, in a straight line with which a set of soundings are to be taken; C is a third known object lying at a sufficient distance to one side of the line A B. The boat is rowed along the straight line B E, either directly towards or directly from B. The surveyor sees that the boatmen keep B and A exactly in one straight line; and the instant that each sounding is taken, he measures with a sextant the angle which the direction of C makes with the line. For example, if 1, 2, 3, 4, &c., are points where soundings are taken, the angles to be measured at those points are B 1 C, B 2 C, B 3 C, B 4 C, &c. The position of C should be so chosen that the most acute of those angles may be 30° or somewhat greater.

Fig. 56.

To plot the positions found by this method, draw through C on the plan the straight line F C D parallel to A B E, and lay off the angles D C 1 = B 1 C, D C 2 = B 2 C, &c.: the intersections of the lines C 1, C 2, &c., with B E, will give the points required.

METHOD II.—*By two Stations and an uniform speed of Rowing.*—In Fig. 57, A represents a known object on which the line of soundings is to run. The surveyor determines the position of (B or C) the commencement of the line by three angles taken between known objects; the rowers then row as steadily as possible at an uniform speed in a straight line directly from or directly towards A. Soundings are taken at equal intervals of time; and when the line has been carried far enough, the surveyor determines the position of its termination (C or B) by three angles taken between known objects.

Fig. 57.

In plotting a line of soundings so taken, its two ends B and C are laid down by means of the station-pointer: the straight line B C is drawn, and divided into as many equal parts as there were equal intervals of time between the soundings from the beginning to the end of the line; and thus the intermediate points, 1, 2, 3, &c., are found.

A line of soundings may often be conveniently prolonged on the higher parts of the beach by ordinary levelling. In fact, levelling should be used wherever it is practicable, being the more accurate operation.

76. The **Reduction of Soundings** to the datum of the survey is made by taking the difference between each sounding and the height of the water above that datum at the instant when the sounding was taken, as found by examination of or interpolation in the register of the tides. If the sounding is the greater, that difference is a depth below the datum,—if the less, a height above the datum. (As to what that datum is, see Article 71). When the datum is the mean low-water-level of spring tides, the latter class of reduced soundings are said to be *dry*, and are distinguished in the register and on the plan by a score beneath the figures

To reduce soundings by calculation, in the absence of direct observations of the tide, it is necessary to know the rise of the tide above the mean water-level, and the time of high-water, for the tide during which the soundings were made, and the *duration* of that tide, or interval of time between high-water and low-water (which on an average is about six hours twelve minutes, but varies considerably at different times and places).

Let H be the height of the mean water-level above the datum:—

r, the rise of the tide above the mean water-level;

D, the duration of the tide;

t, the time before or after high-water at which a given sounding is taken;

h, the height of the surface of the water above the datum at that instant, being the quantity to be subtracted from the sounding.

Then
$$h = H + r \cdot \cos 180^\circ \frac{\quad}{D}, \qquad (1.)$$

in using which formula it is to be remembered that *cosines of obtuse angles are negative.*

77. **Lines of Equal Depth** are analogous to contour-lines on land (see p. 95), being contour-lines of the bottom of the sea sketched on the plan so as to pass through those points where the reduced soundings are equal. It is customary to mark the *line of one fathom soundings* by single dots, of two fathoms by dots in pairs, of three fathoms by dots in triplets, and so on.

The **High** and **Low-Water-Marks** of average spring tides, which should be drawn on the plan, are also analogous to contour-lines.

78. **Currents—Waves.**—The directions and velocities of tidal currents should be noted by the surveyor, and marked on the plan by arrows; each arrow having figures beside it denoting the speed of the current in nautical miles an hour, and the time after the moon's transit at which it prevails. Flood-currents are denoted by feathered arrows, ebb-currents by unfeathered arrows.

The direction of the current which runs past a moored vessel may be ascertained by dropping some floating body into it, and observing the angle which the direction of motion of that body makes with the direction of some known object. The velocity may be found by means of Massey's Log, an instrument in which the rotations of a fan driven by the current are registered by wheel-work.

The direction and velocity of a current may also be determined by setting a light deal pole, having a weight at the lower end, to float upright in it, and taking simultaneous angles to that object from two known stations. This must be done by two observers, who should take special care to make their angular measurements exactly at the same instants of time.

The usual directions and velocities of waves should be ascertained and noted, and also the greatest height from the crest to the trough of a wave.

79. **Miscellaneous Information on Plan.**—Besides the soundings, levels, currents, and other information already mentioned, the plan of a marine survey for engineering purposes should show at different points the material of the bottom, by such abbreviations as *r.* for rock, *st.* for stones, *s.* for sand, *m.* for mud, &c., and by references to borings and examinations by diving, where such have been made. It should also show all lighthouses, beacons, buoys, fixed moorings, &c.

80. **Taking Altitudes by the Sextant—Dip of the Horizon.**—When the altitude of an object is taken at sea by measuring with a sextant its angular elevation above the visible sea-horizon, a correction must be made by subtracting the *dip* of that horizon—that is, its apparent angular depression below a truly horizontal line traversing the eye of the observer. The amount of that depression is uncertain, owing to the variable refractive power of the atmosphere; but on an average, it is given approximately by the following formula, in which h denotes the height of the observer's eye above the sea, and r the radius of curvature of the surface of the sea.

$$\text{Dip in seconds} = \frac{9}{10} \times 206264''\cdot 8 \sqrt{\frac{2h}{r}}$$

$$= 57''\cdot 4 \sqrt{h \text{ in feet}} \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (1.)$$

CHAPTER VII.

OF COPYING, ENLARGING, AND REDUCING PLANS.

81. **Tracing** upon a sheet of thin semi-transparent paper, laid smoothly on the original drawing, is the most accurate method of obtaining a copy of a plan on the same scale with the original. By using a drawing table made of strong plate glass, called the "copying-glass," with a sloping mirror below, if necessary, to reflect light through it, a tracing may be made on drawing paper of ordinary thickness, provided the original is not mounted on cloth.

When a tracing has been made on thin paper, other copies can be made on thick paper by rubbing the lower side of the tracing with black lead, or putting a sheet of black-leaded paper below it, laying it on the thick paper, and passing a smooth pointed instrument along all the outlines of the tracing. The new copy has then to be drawn in ink and finished.

Pricking Through is applicable to plans in which the outlines consist chiefly of straight lines, and damage to the original plan is unimportant.

82. **Engraving, Lithographing, and Printing.**—When a plan is to be engraved on copper, a tracing of it is placed on the copper plate, face downwards, and the outlines scratched on the copper with a point which cuts through the tracing. The impressions from copper plates, being printed on damp paper, shrink when they dry, to an extent which varies from 1-400th to 1-200th of the original dimensions. All measurements, therefore, on printed plans should be made by means of the scale engraved along with the plan, and every sheet should have a scale upon it. The shrinking is sometimes slightly different lengthwise and breadthwise. As to the effect of this on sections, see Article 54, p. 90.

Where great accuracy is required in engraved plans (as in those of the ordnance survey), the principal stations are *plotted on the copper*, and the details only laid down on it by tracing.

In lithographing plans, the usual process is to make a copy on "transfer paper" by the aid of a tracing on thin paper, as already described in the preceding Article. The copy so made is drawn and finished with lithographic ink, laid face downwards on a stone, and transferred to the stone by the proper process. The

transfer paper being damp during that process, expands to a certain extent, so that the drawing on the stone is somewhat larger than the original; and this expansion is to a certain extent counteracted by the shrinking of the paper on which the impressions are printed; so that the impressions may be slightly larger or smaller than the original in a proportion which it is difficult to assign with precision. As with engraved plans, each sheet should have its own scale, by means of which all measurements upon it should be made.

83. **Reducing Drawings by Hand** is performed, in the case of plans, by forming triangles to connect the stations and other principal points on the original, measuring their sides, and plotting them on a smaller scale on the reduced plan. The details may be reduced by covering the original with a network of squares, and the reduced copy with a network of squares having their sides smaller than those of the original squares in the proportion in which the plan is to be reduced, and sketching the details on the reduced copy in their proper places by the aid of those squares to guide the eye and hand.

In the case of sections, reducing by hand is best performed by plotting the section anew on the smaller scale.

84. **Reducing Drawings by Mechanism** is performed by means of instruments called the "Pantograph" and the "Eidograph." In each of those instruments a tracing-point is made to travel over the outlines of the original drawing; a pencil is so connected with the tracing point that it is always in a straight line with the tracing-point, and with a fixed centre, and always at a distance from that centre bearing a given constant ratio to the distance of the tracing-point from that centre, and that pencil draws the outlines of a copy of the drawing reduced in the given ratio.

Fig. 58 is a skeleton sketch of the PANTOGRAPH. F D, D B, E G, and G C are four flat bars, jointed to each other at E, D, C, and G, so that G E = E D = D C = C G, and the figure G E D C is always an exact rhombus, its opposite sides being parallel, and all of them equal. Those bars are supported by ivory castors, which run on the paper or on the drawing-board. T is the tracing point. A is the fixed centre, having a heavy foot, which rests on the paper or the drawing-board. On its vertical spindle turns a socket through which the bar E G can be slid to any required position, and fixed there by a clamp-screw. P is a square socket, sliding on the bar D F, on which it can be fixed in any required position; the pencil is carried by it. The pencil is loaded on the top with weights, which press its point against the paper; it can be lifted off the paper when required by

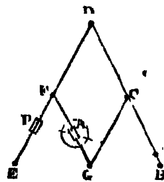


Fig. 58.

pulling a string. The dotted line P A T represents the imaginary straight line in which the pencil, the centre, and the tracing point ought to be situated. The bars G E and E F have scales marked on them, showing the proper positions of the sliders for reducing drawings in various proportions. Let 1 : n be the proportion in which the plan is to be reduced; so that—

$$n : 1 :: \mathrm{T\,A} : \mathrm{A\,P}; \ldots\ldots\ldots (1.)$$

then

$$n : 1 :: \mathrm{D\,E} : \mathrm{E\,P}; \ldots\ldots\ldots (2.)$$

and

$$n + 1 : 1 :: \mathrm{D\,T} : \mathrm{E\,A}. \ldots\ldots\ldots (3.)$$

The **Eidograph** is represented by the skeleton sketch, fig. 59. A is its fixed centre, with a heavy leaden foot. On the spindle of this centre turns a square socket, through which slides the bar D E, which can be clamped in any required position. At the ends of that bar are two pulleys, D and E, exactly equal in diameter, and connected by means of a thin steel belt. F and G are screws for adjusting the lengths of the two divisions of that belt, so as to make the rods B P and T C exactly parallel. These rods slide through square sockets carried by the pulleys, and having clamp-screws. T is the tracing-point, P the pencil, and T A P the imaginary straight line in which the pencil, centre, and tracing-point should always be.

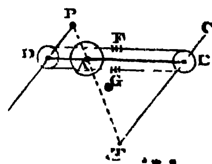


Fig. 59.

Let 1 : n as before be the ratio of reduction; then the proper positions of the sockets are given by the formulæ—

$$n : 1 :: \mathrm{A\,E} : \mathrm{A\,D};\ \mathrm{E\,T} = \mathrm{A\,E};\ \mathrm{D\,P} = \mathrm{A\,D}. \ldots (4.)$$

Each bar has a scale of 200 equal parts on it, with 0 marked at the middle of its length, and numbered to 100 each way. These scales are subdivided by the aid of verniers on the sockets. When the instrument is correctly adjusted, each socket is at the same distance from the middle of its bar; and that distance, in divisions of the scale, is found by the following formula:—

$$m = 100 \cdot \frac{n - 1}{n + 1}. \ldots\ldots\ldots (5.)$$

The best test of the accuracy of the adjustments of the Pantograph and Eidograph is to draw the tracing-point T for a certain distance along the edge of a flat straight-edged ruler; when the pencil P ought to draw an exactly straight line, of a length bearing the proper proportion to the length of the original line.

85. **Enlarging Plans** may be performed by hand, in the same manner with reducing; and with the Pantograph or Eidograph, by adjusting either of those instruments so that the pencil shall be further from the centre than the tracing-point is. This, however, is an operation which is not capable of accuracy, except when the ratio of enlargement does not much exceed that of equality.

86. **Reducing Drawings by Photography** is the method employed in reducing the large plans of the ordnance survey, drawn on a scale of $\frac{1}{2500}$, to the scale of six inches to a mile. The details of the plans so reduced are afterwards traced on the copper plates, on which the stations have been previously plotted by the lengths of the sides of the triangles. A process of transferring the reduced outlines to copper, zinc, or stone, without tracing, has lately been introduced. See the *Report on the Progress of the Ordnance Survey*, by Colonel Sir Henry James, R.E.

Supplement to Chapter III., Article 40.

86 A. **Reduction of Angles to the Centre of the Station.**—It sometimes happens that the theodolite cannot be planted exactly at a station in a trigonometrical survey; but has to be placed at a short distance to one side of it. In such cases, the angle actually measured between two objects is reduced to the angle which would have been measured, had the theodolite been exactly at the station, by a correction which is calculated approximately as follows:—

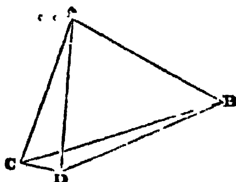


Fig 59 A.

In fig. 59 A, let C be the station, D the position of the theodolite, A and B two objects; A D B the horizontal angle between them as measured at D; A C B the required horizontal angle at the station C.

Measure C D, and the angle A D C; calculate A C and C B approximately as if A C B were equal to A D B; then

$$A\,C\,B = A\,D\,B - 206264''.8\ C\,D\left\{\frac{\sin A\,D\,C}{A\,C} - \frac{\sin B\,D\,C}{B\,C}\right\} \quad (1.)$$

The above formula gives the correction in seconds when D lies to the *right* of both C A and C B. When it lies to the left of C B, sin B D C changes its sign; when to the left of C A, sin A D C changes its sign.

Supplement to Chapter III., Article 42, Division IV., Page 73.

86 B. **Astronomical Refraction.**—The refracting action of the atmosphere causes the altitudes of the stars to appear greater than they really are. The correction for refraction, therefore, is always to be subtracted from an altitude. Its value may be found in seconds approximately by the following formula:—

$$\text{Refraction} = 58'' \times \text{cotan apparent altitude.}$$

For more exact information on the subject, see a paper by the Rev. Dr. Robinson in the *Transactions of the Royal Irish Academy*, vol. xix. Tables of Refraction are given in treatises on Navigation, such as Raper's.

It is to be borne in mind, that below about 8° or 10° of altitude the changeable condition of the atmosphere makes the correction for refraction very uncertain.

86 C. **To find the Latitude of a Place.**

METHOD I.—*By the Mean Altitude of a Circumpolar Star.*—Take the altitudes of a circumpolar star at its upper and lower culmination (which positions are known by watching for the instants when the altitude is greatest and least). From each of those *apparent* altitudes subtract the correction for refraction; the mean of the *true* altitudes thus found is the latitude of the place.

METHOD II.—*By One Meridian Altitude of a Star.*—Observe the meridian altitude of a star by watching for the instant when its altitude is greatest or least, and subtract the corrections for refraction, and also for dip, if necessary. The complement of the true altitude is the *zenith distance.* Find the declination of the star from the *Nautical Almanac* (which is published four years in advance.)*

Then if the star is between the zenith and the equator,

$$\text{Latitude} = \text{Zenith distance} + \text{Declination}; \ldots\ldots \quad (1.)$$

If the star is between the equator and the horizon,

$$\text{Latitude} = \text{Zenith distance} - \text{Declination}; \ldots\ldots \quad (2.)$$

If the star is between the zenith and the elevated pole,

$$\text{Latitude} = \text{Declination} - \text{Zenith distance}; \ldots\ldots \quad (3.)$$

* The declinations of a few stars are given at p. 73.

If the star is between the elevated pole and the horizon,

Latitude = 180° — Declination — Zenith distance. ...(4.)

METHOD III.—*By the Sun's Meridian Altitude.*—In this method the final calculation, from the sun's declination, as found in the *Nautical Almanac*, and the *true* altitude of his centre, is the same as in Method II. But besides the correction for refraction and dip, the altitude requires to be further corrected by subtracting or adding the sun's semidiameter, according as his upper or lower limb has been observed, and by adding the sun's parallax, being the angle subtended at the sun by the distance between the earth's centre and the place of observation.

To find the correction for parallax, find the sun's horizontal parallax on the day of observation, from the *Nautical Almanac*, and multiply it by the cosine of the altitude of the sun's centre.

(The mean value of the sun's horizontal parallax is about 8″·6).

The sun's semidiameter on the day of observation is to be found in the *Nautical Almanac*. It varies from 15′ 46″ to 16′ 18″.

The calculation may be thus set down algebraically—

True altitude = apparent altitude — Dip (if the sea-horizon has been observed) — Refraction ± sun's semidiameter + parallax; .... (5.)

Zenith distance = 90° — true altitude, ... (6.)

Latitude (see Equations 1, 2, 3, 4).

Equations 1 and 2 are the most frequently applicable to the sun. Equation 3 is occasionally applicable between the tropics; and Equation 4 relates to observations made at midnight, in summer, in the polar regions.

86 D. **List of Authorities on Engineering Geodesy and Subjects connected with it.**—Merrett's *Surveying*; Jackson's *Survey Practice*; Gurden's *Traverse Tables*; Brough's *Mine-Surveying*; Butler Williams's *Practical Geodesy*; Haskoll's *Engineering Field-Work*; Haskoll *On Railway Construction*; Simms *On Mathematical Instruments*, Simms *On Levelling*; Simms's *Practical Tunnelling*; Sir Edward Belcher *On Marine Surveying*; *Admiralty Manual of Scientific Inquiry*, Article "Hydrography;" Raper's *Navigation*.

PART II.

OF MATERIALS AND STRUCTURES.

CHAPTER I.

SUMMARY OF PRINCIPLES OF STABILITY AND STRENGTH.

SECTION 1.—*Of Structures in General.*

87. A **Structure** consists of portions of solid materials, put together so as to preserve a definite form and arrangement of parts, and to withstand external forces tending to disturb such form and arrangement. As the parts of a structure are intended to remain at rest relatively to each other, the forces which act on the whole structure, and on each of its parts, should be *balanced*, so that the mechanical principles on which the permanence and efficiency of structures depend for the most part belong to STATICS, or the science of balanced forces.

The *materials* of a structure may be more or less stiff, like stone, timber, and metals, or loose, like earth.

The ensuing chapters of this part will be divided according to the materials of which the structures they treat of consist. In the present chapter are given a summary of mechanical principles applicable to all structures. Many passages in it are extracted from a previous Treatise on *Applied Mechanics*, and abridged or amplified as may be required, in order to suit the purpose of the present Treatise. Such passages are indicated by the letters *A. M.*, with a reference to the number of the corresponding Article in that work.

87 A. **Pieces—Joints—Supports—Foundations.** (*A. M.*, 129, 130).—A structure consists of two or more solid bodies, called its *pieces*, which touch each other and are connected at portions of their surfaces called *joints*. This statement may appear to be applicable to structures of stiff materials only; but, nevertheless, it comprehends masses of earth also, if they are considered as consisting of a very great number of very small pieces, touching each other at innumerable joints.

Although the pieces of a structure are fixed relatively to each

other, the structure as a whole may be either fixed or moveable relatively to the earth.

A fixed structure is supported on a part of the solid material of the earth, called the *foundation* of the structure; the pressures by which the structure is supported, being the resistances of the various parts of the foundation, may be more or less oblique.

A moveable structure may be supported, as a ship, by floating in water, or as a carriage, by resting on the solid ground through wheels. When such a structure is actually in motion, it partakes to a certain extent of the properties of a machine; and the determination of the forces by which it is supported requires the consideration of dynamical as well as of statical principles; but when it is not in actual motion, though capable of being moved, the pressures which support it are determined by the principles of statics; and it is obvious that they have their resultant equal and directly opposed to the weight of the structure.

88. **The Conditions of Equilibrium of a Structure** are the three following (*A. M.*, 131):—

I. *That the forces exerted on the whole structure by external bodies shall balance each other.*—The forces to be considered under this head are—(1.) the *Attraction of the Earth*—that is, the *weight* of the structure; (2.) the *External Load*, arising from the pressures exerted against the structure by bodies not forming part of it nor of its foundation; (these two kinds of forces constitute the *gross* or *total load*); (3.) the *Supporting Pressures*, or resistance of the foundation. These three classes of forces will be spoken of together as the *External Forces*.

II. *That the forces exerted on each piece of the structure shall balance each other.*—These consist of—(1.) the *Weight* of the piece, and (2.) the *External Load* on it, making together the *Gross Load*; and (3.) the *Resistances*, or forces exerted at the joints, between the piece under consideration and the pieces in contact with it.

III. *That the forces exerted on each of the parts into which each piece of the structure can be conceived to be divided shall balance each other.*—Suppose an ideal surface to divide any part of any one of the pieces of the structure from the remainder of the piece; the forces which act on the part so considered are—(1.) its weight, and (2.) (if it is at the external surface of the piece) the external force applied to it, if any, making together its *gross load*, (3.) the *stress*, or force, exerted at the ideal surface of division, between the part in question and the other parts of the piece.

89. **Stability, Strength, and Stiffness.** (*A. M.*, 132, 127).—It is necessary to the permanence of a structure, that the three foregoing conditions of equilibrium should be fulfilled, not only under one amount and one mode of distribution of load, but under all the

variations of the load as to amount and mode of distribution which can occur in the use of the structure.

Stability consists in the fulfilment of the *first* and *second* conditions of equilibrium of a structure under all variations of the load within given limits. A structure which is deficient in stability gives way by the displacement of its pieces from their proper positions.

When a structure, or one of its parts, is *flexible*, like the chain of a suspension bridge, or in any other way free to move, its stability consists in a tendency to recover its original figure and position after having been disturbed.

Strength consists in the fulfilment of the *third* condition of equilibrium of a structure for all loads not exceeding prescribed limits; that is to say, the greatest internal stress produced in any part of any piece of the structure, by the prescribed greatest load, must be such as the material can bear, not merely without immediate breaking, but without such injury to its texture as might endanger its breaking in the course of time.

A piece of a structure may be rendered unfit for its purpose, not merely by being broken, but by being stretched, compressed, bent, twisted, or otherwise strained out of its proper shape. It is necessary, therefore, that each piece of a structure should be of such dimensions that its alteration of figure under the greatest load applied to it shall not exceed given limits. This property is called *stiffness*, and is so connected with strength that it is necessary to consider them together.

SECTION II.—*Summary of the Principles of the Balance of Forces.*

90. (*A. M.*, 12, 13, 17 to 24).—A **Force** is an action between two bodies, either causing or tending to cause change in their relative rest or motion. **Equilibrium** or **Balance** is the condition of two or more forces which are so opposed that their combined action on a body produces no change in its rest or motion, and that each force merely *tends* to cause such change, without actually causing it.

In treatises on statics, the word *pressure* is often used to denote any balanced force; although, in the popular sense, that word is used to denote a force, of the nature of a thrust or push, distributed over a surface.

The relation of a force to one of the two bodies between which it acts, is determined, or made known, when the following three things are known respecting it:—first, the *place*, or part of the body to which it is applied; secondly, the *direction* of its action; thirdly, its *magnitude*.

I. The *place of the application* of a force to a body may be the whole or part of its internal mass; in which case the force is an *attraction* or a *repulsion*, according as it tends to move the bodies between which it acts towards or from each other; or the place of application may be the surface at which two bodies touch each other, or the bounding surface between two parts of the same body, in which case the force is a *tension or pull*, a *thrust or push*, or a *lateral stress*, according to circumstances.

Thus every force has its action distributed over a certain space, either a volume or a surface; and a force concentrated at a single point has no real existence. Nevertheless, it is necessary, in treating of the principles of statics, to begin by demonstrating the properties of such ideal forces, conceived to be concentrated at single points; for the conclusions so arrived at respecting *single forces* (as they may be called), are applicable to the distributed forces which really act in nature.

In reasoning respecting forces concentrated at single points, they are assumed to be applied to solid bodies which are *perfectly rigid*, or incapable of alteration of figure under any forces which can be applied to them. This also is a supposition not realized in nature; but its consequences may be applied to actual bodies, when their alterations of figure are insensible.

II. The *direction* of a force is that of the motion which it tends to produce. A straight line drawn through the point of application of a single force, and along its direction, is the *line of action* of that force.

III. The *magnitudes* of two forces are equal, when, being applied to the same body in opposite directions along the same line of action, they balance each other.

A single force may be represented on paper by an arrow-headed straight line; the commencement of the line indicating the point of application of the force,—the direction of the line, the direction of the force,—and the length of the line, the magnitude of the force, according to an arbitrary scale.

91. **Standard Unit of Weight.** (*A. M.*, 21).—The magnitude of a force is expressed arithmetically by stating in numbers its ratio to a certain *unit* or *standard* of force, which is usually the *weight* (or attraction towards the earth), at a certain latitude, and at a certain level, of a known mass of a certain material. Thus the British unit of force is the *standard pound avoirdupois;* which is the weight in the latitude of London, and near the level of the sea, of a certain piece of platinum kept in the Exchequer office. (See the Act 18 and 19 Vict., cap. 72; also a paper by Professor W. H. Miller, in the *Philosophical Transactions* for 1856.)

Amongst other units of force employed in Britain are,—

The grain = $\frac{1}{7000}$ of a pound avoirdupois.

The troy pound = 5,760 grains = $0 \cdot 82\dot{2}85714\dot{}$ pound avoirdupois.

The hundredweight = 112 pounds avoirdupois.

The ton = 2,240 pounds avoirdupois.

The French standard unit of force is the *gramme*, which is the weight, in the latitude of Paris, of a cubic centimetre of pure water, measured at the temperature at which the density of water is greatest, viz., 3°·945 centigrade, or 39°·1 Fahrenheit, and under the pressure which supports a barometric column of 760 millimetres of mercury—that is, 29 922 inches.

A comparison of French and British measures of force and of size is given in a table at the end of this volume.

92. **Resultant of Forces Acting in One Straight Line.** (*A. M.*, 22).—The RESULTANT of any number of given forces applied to one body, is a single force capable of balancing that single force which balances the given forces; that is to say, the resultant of the given forces is equal and directly opposed to the force which balances the given forces; and is *equivalent* to the given forces so far as the balance of the body is concerned. The given forces are called *components* of their resultant.

The resultant of a set of balanced forces is nothing.

The resultant of any number of forces acting on one body in the same straight line of action, acts along that line, and is equal in magnitude to the sum of the component forces; it being understood, that when some of the component forces are opposed to the others, the word "*sum*" is to be taken in the algebraical sense; that is to say, that forces acting in the same direction are to be added to, and forces acting in opposite directions subtracted from each other.

When a system of forces acting along one straight line are balanced, the sum of the forces acting in one direction is equal to the sum of the forces acting in the opposite direction.

93. **Resultant and Balance of Inclined Forces.** (*A. M.*, 51 to 54).—The smallest number of inclined forces which can balance each other is three. Those three forces must act through one point, and in one plane. Their relation to each other depends on the following theorem, called the "PARALLELOGRAM OF FORCES," from which the whole science of statics may be deduced.

I. *If two forces whose lines of action traverse one point be represented in direction and magnitude by the sides of a parallelogram, their resultant is represented by the diagonal.*

For example, through the point O (fig. 60) let two forces act, represented in direction and magnitude by $\overline{OA}$ and $\overline{OB}$. The resultant or equivalent single force of those two forces is represented in direction and magnitude by the diagonal O C of the parallelogram O A C B. Its magnitude is given algebraically by the equation,

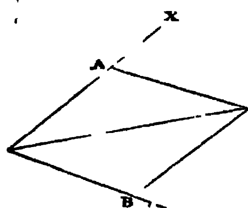


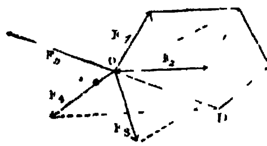
Fig. 60

$$OC = \sqrt{\{OA^2 + OB^2 + 2 \cdot OA \cdot OB \cos AOB\}} \quad (1.)$$

To balance the forces $\overline{OA}$ and O B, a force is required equal and directly opposed to their resultant $\overline{OC}$. This may be expressed by saying, that *if the directions and magnitudes of three forces be represented by the three sides of a triangle* (such as $\overline{OA}$, $\overline{AC}$, $\overline{CO}$), *then those three forces, acting through one point, balance each other*, or in other words, that three forces in the same plane balance each other at one point, when each is proportional to the sine of the angle between the other two

The following corollary from the parallelogram of forces is called the "POLYGON OF FORCES":—

II. *If a number of forces acting through the same point be represented by lines equal and parallel to the sides of a closed polygon, those forces balance each other.* To fix the ideas, let there be five forces acting through the point O (fig 61), and represented in direction and magnitude by the lines F_1, F_2, F_3, F_4, F_5, which are equal and parallel to the sides of the closed polygon O A B C D O; viz. :—


Fig. 61.

$$F_1 = \text{and} \parallel OA; \; F_2 = \text{and} \parallel AB; \; F_3 = \text{and} \parallel BC;$$

$$F_4 = \text{and} \parallel CD; \; F_5 = \text{and} \parallel DO$$

Then, by the principle of the parallelogram of forces, the resultant of F_1 and F_2 is O B; the resultant of F_1, F_2, and F_3 is O C; the resultant of F_1, F_2, F_3, and F_4 is O D, equal and opposite to F_5, so that the final resultant is nothing.

The closed polygon may be either plane or "gauche"—that is, not in one plane.

III. *Principle of the Parallelopiped of Forces.*—The simplest gauche polygon is one of four sides. Let O A B C E F G H (fig. 62), be a parallelopiped whose diagonal is O H. Then any three successive edges so placed as to begin at O and end at H, form, together with the diagonal H O, a closed quadrilateral; consequently, if three forces F_1, F_2, F_3, acting through O, be represented by the three edges O A, O B, $\overline{O}$ C, of a parallelopiped, the diagonal O $\overline{H}$ represents their resultant, and a fourth force F_4 equal and opposite to O H balances them.

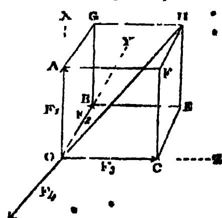


Fig. 62.

94. **Resolution of a Force.**—I. *Into two Components.* (*A. M.*, 55, 56).—In order that a given single force may be resolvable into two components acting in given lines inclined to each other, it is necessary, *first*, that the lines of action of those components should intersect the line of action of the given force in one point; and *secondly*, that those three lines of action should be in one plane.

Returning then to Fig 60, let O C represent the given force, which it is required to resolve into two component forces, acting in the lines O X, O Y, which lie in one plane with O C, and intersect it in one point O.

Through C draw C A ∥ O Y, cutting O X in A, and C B ∥ O X, cutting O Y in B Then will O A and O B represent the component forces required.

Two forces respectively equal to and directly opposed to O $\overline{A}$ and $\overline{O}$ B will balance O C.

The magnitudes of the forces are in the following proportions:—

O C : O A : O B

: : sin A O B : sin B O C : sin A O C... ....(1.)

II. *Into three Components.*—In order that a given single force may be resolvable into three components acting in given lines inclined to each other, it is necessary that the lines of action of the components should intersect the line of action of the given force in one point.

Returning to Fig. 62, let $\overline{O}$ H represent the given force which it is required to resolve into three component forces, acting in the lines O X, O Y, O Z, which intersect O H in one point O.

Through H draw three planes parallel respectively to the planes Y O Z, Z O X, X O Y, and cutting respectively O X in A, O Y in B, O Z in C. Then will $\overline{O}$ A, $\overline{O}$ B, $\overline{O}$ C, represent the component forces required.

Three forces respectively equal to, and directly opposed to $\overline{OA}$, $\overline{OB}$, and $\overline{OC}$, will balance $\overline{OH}$.

III. *Rectangular Components.*—The rectangular components of a force are those into which it is resolved when the directions of their lines of action are at right angles to each other.

For example, in fig. 62, suppose O X, O Y, O Z, to be three axes of co-ordinates at right angles to each other. Then $\overline{OH}$ is resolved into three rectangular components, O A, O B, O C, simply by letting fall from H perpendiculars on O X, O Y, O Z, cutting them at A, B, C, respectively.

Let the three rectangular components be denoted respectively by X, Y, Z, the resultant by R, and the angles which it makes with the components by α, β, γ, respectively; then the relations between the three rectangular components and their resultant are expressed by the following equations:—

$$X = R \cos \alpha; \quad Y = R \cos \beta; \quad Z = R \cos \gamma; \quad \ldots\ldots\ldots (2.)$$

$$R^2 = X^2 + Y^2 + Z^2. \quad \ldots\ldots\ldots\ldots\ldots (3.)$$

When the resultant is in the same plane with two of its components (as X and Y), the third component is null, and the equations 2 and 3 take the following form:—

$$X = R \cos \alpha = R \sin \beta; \quad Y = R \cos \beta = R \sin \alpha; \quad Z = 0; \ldots (4.)$$

$$R^2 = X^2 + Y^2 \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (5.)$$

In using equations 2, 3, 4, and 5, it is to be remembered that cosines of obtuse angles are negative.

95. **Resultant and Balance of any number of Inclined Forces Acting through one Point.**—To find this resultant by calculation, assume any three directions at right angles to each other as axes; resolve each force into three components (X, Y, Z) along those axes, and consider the components along a given axis which act in one direction as positive, and those which act in the opposite direction as negative; take the algebraical sums of the components along the three axes respectively ($\Sigma \cdot X$, $\Sigma \cdot Y$, $\Sigma \cdot Z$); these will be the *rectangular components of the resultant of all the forces;* and its magnitude and direction will be given by the following equations:—

$$R^2 = (\Sigma \cdot X)^2 + (\Sigma \cdot Y)^2 + (\Sigma \cdot Z)^2; \ldots\ldots\ldots\ldots (1.)$$

$$\cos \alpha = \frac{\Sigma \cdot X}{R}; \quad \cos \beta = \frac{\Sigma \cdot Y}{R}; \quad \cos \gamma = \frac{\Sigma \cdot Z}{R}. \quad \ldots\ldots (2.)$$

If the forces all act in one plane, two rectangular axes in that

plane are sufficient, and the terms containing Z disappear from the equations.

If the forces balance each other, the components parallel to each axis balance each other independently; that is to say, the three following conditions are fulfilled:—

$$\Sigma \cdot X = 0; \ \Sigma \cdot Y = 0; \ \Sigma \cdot Z = 0 \ldots\ldots\ldots\ldots\ldots (3.)$$

If the forces all act in one plane, these *conditions of equilibrium* are reduced to two.

96. **Resultant and Balance of Couples.** (*A. M.*, 25 to 37).—Two forces of equal magnitude applied to the same body in parallel and opposite directions, but not in the same line of action (such as F, F, in fig. 63), constitute what is called a "*couple.*"

The *arm* or *leverage* of a couple (L, fig. 63) is the perpendicular distance between the lines of action of the two equal forces.

The tendency of a couple is to turn the body to which it is applied in the *plane* of the couple—that is, the plane which contains the lines of action of the two forces. (The plane in which a body turns is any plane parallel to those planes in the body whose position is not altered by the turning). The turning of a body is said to be *right-handed* when it appears to a spectator to take place in the same direction with that of the hands of a watch, and *left-handed* when in the opposite direction; and couples are designated as right-handed or left-handed according to the direction of the turning which they tend to produce. The couple represented in fig. 63 appears right-handed to the reader.

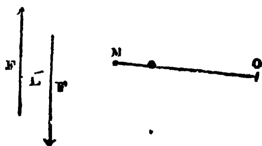


Fig. 63.

The *Moment* of a couple means the product of the magnitude of its force by the length of its arm (F L); and may be represented by the area of a rectangle whose sides are F and L. If the force be a certain number of pounds, and the arm a certain number of feet, the product of those two numbers is called the moment in *foot-pounds*, and similarly for other measures. The moment of a couple may also be represented by a single line on paper, by setting off upon its *axis* (that is, upon any line perpendicular to the plane of the couple) a length proportional to that moment (O M, fig. 63) in such a direction, that to an observer looking from O towards M the couple shall seem right-handed.

The following principle is the groundwork of the theory of couples. It may also be made the groundwork of the whole science of statics, instead of the principle of the parallelogram of forces; for each of those two principles is a necessary consequence of the other.

I. *If the moments of two couples acting in the same direction and in the same or parallel planes are equal, those couples are equivalent:* that is, their tendencies to turn the body to which they are applied are the same.

The following propositions are the chief consequences of the principle just stated.

II. The resultant of any number of couples acting in the same or parallel planes is equivalent to a couple whose moment is the algebraical sum of the moments of the combined couples.

III. Two opposite couples of equal moment in the same or parallel planes balance each other. Any number of couples in the same or parallel planes balance each other when the moments of the right-handed couples are together equal to the moments of the left-handed couples; in other words, when the resultant moment is nothing—a condition expressed algebraically by

$$\Sigma \cdot F L = 0. \quad \ldots\ldots\ldots\ldots \quad (1.)$$

IV. If the two sides of a parallelogram represent the axes and moments of two couples acting on the same body in planes inclined to each other, the diagonal of the parallelogram will represent the axis and moment of the resultant couple, which is equivalent to those two.

In other words, three couples represented by the three sides of a triangle balance each other.

V. If any number of couples acting on the same body be represented by a series of lines joined end to end, so as to form sides of a polygon, and if the polygon is closed, those couples balance each other.

These propositions are analogous to corresponding propositions relating to single forces, and couples, like single forces, can be resolved into components acting about two or three given axes.

97. **Resultant and Balance of Parallel Forces.** (*A. M.*, 38 to 47). —A balanced system of parallel forces consists either of pairs of directly opposed equal forces, or of couples of equal forces, or of combinations of such pairs and couples.

Hence the following propositions as to the relations amongst the *magnitudes* of systems of parallel forces.

I. In a balanced system of parallel forces the sums of the forces acting in opposite directions are equal; in other words, the algebraical sum of the magnitudes of all the forces taken with their proper signs is nothing.

II. The magnitude of the resultant of any combination of parallel forces is the algebraical sum of the magnitudes of the forces.

The relations amongst the *positions* of the lines of action of balanced parallel forces remain to be shown; and in this inquiry

all pairs of directly opposed equal forces may be left out of consideration; for each such pair is independently balanced whatsoever its position may be; so that the question in each case is to be solved by means of the theory of couples.

The following is the simplest case:—

III. **Principle of the Lever.**—*If three parallel forces applied to one body balance each other, they must be in one plane; the two extreme forces must act in the same direction; the middle force must act in the opposite direction; and the magnitude of each force must be proportional to the distance between the lines of action of the other two.* Let a body (fig. 64) be maintained in equilibrio by two opposite couples acting in the same plane, and of equal moments,

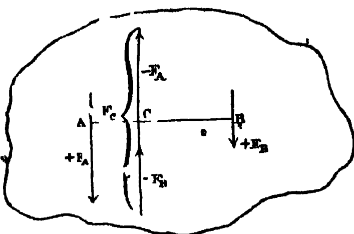


Fig. 64

$$F_A L_A \quad F_B L_B,$$

and let those couples be so applied to the body that the lines of action of two of those forces, $-F_A$ $-F_B$, which act in the same direction, shall coincide. Then those two forces are equivalent to the single middle force $F_C = -(F_A + F_B)$, equal and opposite to the sum of the extreme forces $+F_A$, $+F_B$, and in the same plane with them, and if the straight line A C B be drawn perpendicular to the lines of action of the forces, then

$$\overline{AC} = L_A, \ \overline{CB} = L_B; \ \overline{AB} = L_A + L_B;$$

and consequently

$$F_A : F_B : F_C :: \overline{CB} : \overline{AC} : \overline{AB}; \qquad (1.)$$

This proposition holds also when the straight line A C B crosses the lines of action of the three forces obliquely.

IV. The *resultant* of any two of the three forces F_A, F_B, F_C, is equal and opposite to the third.

In order that two opposite parallel forces may have a single resultant, it is necessary that they should be unequal, the resultant being their difference. Should they be equal, they constitute a couple, which has no single resultant.

V. *Resultant of a Couple and a Single Force in Parallel Planes.*—

Let M denote the moment of a couple applied to a body (fig. 65); and at a point O let a single force F be applied, in a plane parallel to that of the couple. For the given couple substitute an equivalent couple, consisting of a force − F equal and directly opposed to F at O, and a force F acting through the point A, the arm $\overline{AO}$ perpendicular to F being $= \frac{M}{F}$, and parallel to the plane of the couple M. Then the forces at O balance each other, and F acting through A is the resultant of the single force F applied at O, and the couple M; that is to say, that if with a single force F there be combined a couple M whose plane is parallel to the force, the effect of that combination is to shift the line of action of the force parallel to itself through a distance $OA = \frac{M}{F}$,—to the left if M is right-handed—to the right if M is left-handed.

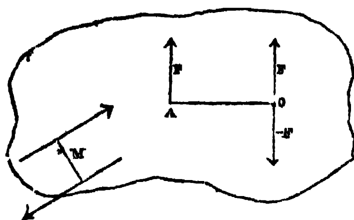


Fig. 65.

VI. *Moment of a Force with respect to an Axis.*—In fig. 66, let the straight line F represent a force. Let O X be any straight line perpendicular in direction to the line of action of the force, and not intersecting it, and let A B be the common perpendicular of those two lines. At B conceive a pair of equal and directly opposed forces to be applied in a line of action parallel to F, viz.: F′ = F, and − F′ = − F. The supposed application of such a pair of balanced forces does not alter the statical condition of the body. Then the original single force F, applied in a line traversing A, is equivalent to the force F′ applied in a line traversing B, the point in O X which is nearest to A, combined with the couple composed of F and − F′, whose moment is F · A B. This is called the *moment of the force* F *relatively to the axis* O X, and sometimes also, the *moment of the force* F *relatively to the plane* traversing O X, parallel to the line of action of the force.

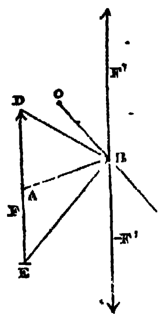


Fig. 66.

If from the point B there be drawn two straight lines B D and B E, to the extremities of the line F representing the force, the area of the triangle B D E being $= \frac{1}{2}$ F · $\overline{AB}$, represents one-half of the moment of F relatively to O X.

VII. *Balance of any System of Parallel Forces in One Plane.*—

In order that any system of parallel forces whose lines of action are in one plane may balance each other, it is necessary and sufficient that the following conditions should be fulfilled:—

First—(As already stated) that the algebraical sum of the forces shall be nothing:—

Secondly—That the algebraical sum of the moments of the forces relatively to any axis perpendicular to the plane in which they act shall be nothing,

two conditions which are expressed symbolically as follows:— Let F denote any one of the forces, considered as positive or negative, according to the direction in which it acts; let y be the perpendicular distance of the line of action of this force from an arbitrarily assumed axis O X, y also being considered as positive or negative, according to its direction; then,

$$\Sigma \cdot F = 0;\ \Sigma \cdot y\, F = 0 \ \ldots\ldots\ldots\ldots\ldots\ \ldots (2.)$$

In summing moments, right-handed couples are usually considered as positive, and left handed couples as negative.

VIII. Let R denote the *resultant of any number of parallel forces in one plane*, and y_r, the distance of the line of action of that resultant from the assumed axis O X to which the positions of forces are referred; then,

$$R = \Sigma \cdot F;$$

$$y_r = \frac{\Sigma \cdot y\, F}{\Sigma \cdot F}.$$

In some cases, the forces may have no single resultant, $\Sigma \cdot F$ being $= 0$; and then, unless the forces balance each other completely, their resultant is a couple of the moment $\Sigma \cdot y\, F$.

IX. *Balance of any System of Parallel Forces.*—In order that any system of parallel forces, whether in one plane or not, may balance each other, it is necessary and sufficient that the three following conditions should be fulfilled:—

First—(As already stated) that the algebraical sum of the forces shall be nothing:—

Secondly and Thirdly—That the algebraical sums of the moments of the forces, relatively to a pair of axes at right angles to each other, and to the lines of action of the forces, shall each be nothing,

two conditions which are expressed symbolically as follows:— Let O X and O Y denote the pair of axes; let F be the magnitude of any one of the forces; y its perpendicular distance from O X, and x its perpendicular distance from O Y; then,

$$\Sigma \cdot F = 0;\ \Sigma \cdot y\, F = 0;\ \Sigma \cdot x\, F = 0; \ldots\ldots\ldots\ldots\ldots (3.)$$

X. Let R denote the *resultant of any system of parallel forces*, and x_r and y_r the distances of its line of action from two rectangular axes; then,

$$R = \Sigma \cdot F; \; x_r = \frac{\Sigma \cdot x F}{\Sigma \cdot F}; \; y_r = \frac{\Sigma \cdot y F}{\Sigma \cdot F} \quad \text{............(4.)}$$

In some cases the forces may have no single resultant, $\Sigma \cdot F$ being $= 0$; and then, unless the forces balance each other completely, their resultant is a couple, whose axis, direction, and moment, are found as follows:—

Let $$M_x = \Sigma . y F; \; M_y = -\Sigma . x F;$$

be the moments of the pair of partial resultant couples about the axes O X and O Y respectively. From O, along those axes, set off two lines representing respectively M_x and M_y: that is to say, proportional to those moments in length, and pointing in the direction from which those couples must respectively be viewed in order that they may appear right-handed. Complete the rectangle whose sides are those lines; its diagonal will represent the axis, direction, and moment of the final resultant couple. Let M_r be the moment of this couple; then,

$$M_r = \sqrt{\left\{ M_x^2 + M_y^2 \right\}}, \quad \text{..................(5.)}$$

and if θ be the angle which its axis makes with O X,

$$\cos \theta = \frac{M_x}{M_r} \quad \text{.. ....(6.)}$$

98. The **Centre of Parallel Forces** (*A. M.*, 49, 50) is the single point referred to in the following principle. The forces to which that principle is applied are in general either weights or pressures; and the point in question is then called the *Centre of Gravity* or the *Centre of Pressure*, as the case may be.

If there be given a system of points, and the mutual ratios of a system of parallel forces applied to those points, which forces have a single resultant, then there is one point, and one only, which is traversed by the line of action of the resultant of every system of parallel forces having the given mutual ratios and applied to the given system of points, whatsoever may be the absolute magnitudes of those forces, and the angular position of their lines of action.

The position of that point is found as follows:—

Let O in fig. 67 be any convenient point, taken as the origin of co-ordinates, and O X, O Y, O Z, three axes of co-ordinates at right angles to each other,

Let A be any one of the points to which the system of parallel forces in question are applied. From A draw x parallel to O X, and perpendicular to the plane Y Z, y parallel to O Y, and perpendicular to the plane Z X, and z parallel to O Z, and perpendicular to the plane X Y. x, y, and z are the rectangular co-ordinates of A, which, being known, the position of A is determined. Let F denote either the magnitude of the force applied at A, or any magnitude proportional to that magnitude. x, y, z, and F are supposed to be known for every point of the given system of points.

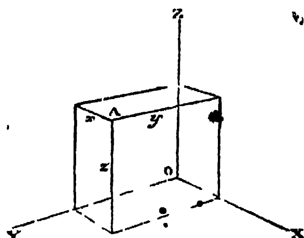


Fig 67.

First, conceive all the parallel forces to act in lines parallel to the plane Y Z. Then the distance of their resultant, and of the centre of parallel forces from that plane is

$$x_r = \frac{\Sigma \cdot x F}{\Sigma \cdot F}. \qquad (1.)$$

Secondly, conceive all the parallel forces to act in lines parallel to the plane Z X. Then the distance of their resultant, and of the centre of parallel forces from that plane is

$$y_r = \frac{\Sigma \cdot y F}{\Sigma \cdot F}. \qquad (2.)$$

Thirdly, conceive all the parallel forces to act in lines parallel to the plane X Y. Then the distance of their resultant, and of the centre of parallel forces from that plane, is

$$z_r = \frac{\quad}{F}. \qquad (3.)$$

If the forces have no single resultant, so that $\Sigma \cdot F = 0$, there is no centre of parallel forces. This may be the case with pressures, but not with weights.

If the parallel forces applied to a system of points are all equal and in the same direction, it is obvious that the distance of the centre of parallel forces from any given plane is simply the mean of the distances of the points of the system from that plane.

99. **Resultant and Balance of any System of Forces in One Plane.** (*A. M.*, 59).—Let the plane be that of the axes O X and O Y in fig. 67; and in looking from Z towards O, let Y lie to the right of

X, so that rotation from X towards Y shall be right-handed. Let x and y be the co-ordinates of the point of application of one of the forces, or of any point in its line of action, relatively to the assumed origin and axes. Resolve each force into two rectangular components X and Y, as in Article 94, p. 137; then the rectangular components of the resultant are $\Sigma \cdot X$ and $\Sigma \cdot Y$; its magnitude is given by the equation

$$R^2 = (\Sigma \cdot X)^2 + (\Sigma \cdot Y)^2; \quad \text{.......... ....} \quad (1.)$$

and the angle α_r which it makes with O X is found by the equations

$$\cos \alpha_r = \frac{\Sigma \cdot X}{R}; \ \sin \alpha_r = \frac{\Sigma \cdot Y}{R}. \quad \text{..............} \quad (2.)$$

This angle is acute or obtuse according as $\Sigma \cdot X$ is positive or negative; and it lies to the right or left of O X according as $\Sigma \cdot Y$ is positive or negative.

The resultant moment of the system of forces about the axis O Z is

$$M = \Sigma (x Y - y X), \quad \text{........ ....} \quad (3.)$$

and is right or left-handed according as M is positive or negative.

The perpendicular distance of the resultant force R from O is

$$L = \frac{M}{R}. \quad \text{....... ....} \quad (4.)$$

Let x_r and y_r be the co-ordinates of any point in the line of action of that resultant; then the equation of that line is

$$x_r \Sigma \cdot Y - y_r \Sigma \cdot X = M. \quad \text{............ .....} \quad (5.)$$

If $M = 0$, the resultant acts through the origin O; if M has magnitude, and $R = 0$ (in which case $\Sigma \cdot X = 0$, $\Sigma \cdot Y = 0$) the resultant is a couple. The conditions of equilibrium of the system of forces are

$$\Sigma \cdot X = 0; \ \Sigma \cdot Y = 0; \ M = 0. \quad \text{. ....} \quad (6.)$$

100. **Resultant and Balance of any System of Forces.** (*A. M.*, 60.) —To find the resultant and the conditions of equilibrium of any system of forces acting through any system of points, the forces and points are to be referred to three rectangular axes of co-ordinates.

As before, let O in fig. 67, p. 145, denote the origin of co-ordinates, and O X, O Y, O Z, the three rectangular axes; and let them be arranged so that in looking from

$$\left.\begin{matrix} X \\ Y \\ Z \end{matrix}\right\} \text{towards O, rotation from} \left\{\begin{matrix} Y \text{ towards } Z \\ Z \text{ towards } X \\ X \text{ towards } Y \end{matrix}\right\}$$

shall appear right-handed.

Let X, Y, Z, denote the rectangular components of any one of the forces; x, y, z, the co-ordinates of a point in its line of action.

Taking the algebraical sums of all the forces which act along the same axes, and of all the couples which act round the same axes, the six following quantities are found, which compose the resultant of the given system of forces:—

Forces.

$$\Sigma \cdot X, \ \Sigma \cdot Y; \ \Sigma \cdot Z, \ \ldots\ldots\ldots\ldots \quad (1.)$$

Couples.

$$\left.\begin{matrix} \text{about O X; } M_1 = \Sigma\,(yZ - zY); \\ \text{,, O Y, } M_2 = \Sigma\,(zX - xZ); \\ \text{,, O Z; } M_3 = \Sigma\,(xY - yX). \end{matrix}\right\} \ldots\ldots\ldots \quad (2.)$$

The three forces are equivalent to a single force

$$R = \sqrt{\left\{(\Sigma \cdot X)^2 + (\Sigma \cdot Y)^2 + (\Sigma \cdot Z)^2\right\}}, \ \ldots\ldots \quad (3.)$$

acting through O in a line which makes with the axes the angles given by the equations

$$\cos \alpha = \frac{\Sigma \cdot X}{R}; \ \cos \beta = \frac{\Sigma \cdot Y}{R}; \ \cos \gamma = \frac{\Sigma \cdot Z}{R}. \ \ldots\ldots \quad (4.)$$

The three couples M_1, M_2, M_3, are equivalent to one couple, whose magnitude is given by the equation

$$M = \sqrt{(M_1^2 + M_2^2 + M_3^2)}, \ \ldots\ldots\ldots\ldots \quad (5.)$$

and whose axis makes with the axes of co-ordinates the angles given by the equations

$$\cos \lambda = \frac{M_1}{M}; \ \cos \mu = \frac{M_2}{M}; \ \cos \nu = \frac{M_3}{M}, \ \ldots\ldots\ldots \quad (6.)$$

in which $\left\{\begin{matrix} \lambda \\ \mu \\ \nu \end{matrix}\right\}$ denote respectively the angles made by the axis of M with $\left\{\begin{matrix} O X \\ O Y \\ O Z \end{matrix}\right\}$.

The *Conditions of Equilibrium* of the system of forces may be expressed in either of the two following forms:—

$$\Sigma \cdot X = 0;\ \Sigma \cdot Y = 0;\ \Sigma \cdot Z = 0;\ M_x = 0;\ M_y = 0;\ M_z = 0; \quad (7.)$$

or

$$R = 0;\ M = 0 \ \ldots\ldots\ldots\ldots \ (8.)$$

When the system is not balanced, its resultant may fall under one or other of the following cases:—

CASE I.—*When* $M = 0$, the resultant is the single force R acting through O.

CASE II.—*When the axis of* M *is at right angles to the direction of* R,—a case expressed by the following equation:—

$$\cos \alpha \cos \lambda + \cos \beta \cos \mu + \cos \gamma \cos \nu = 0; \ \ldots\ldots \ (9.)$$

the resultant of M and R is a single force equal and parallel to R, acting in a plane perpendicular to the axis of M, and at a perpendicular distance from O given by the equation

$$L = \frac{M}{R} \ \ldots\ldots\ldots\ldots\ldots\ldots \ (10.)$$

CASE III.—*When* $R = 0$, there is no single resultant; and the only resultant is the couple M.

CASE IV.—*When the axis of* M *is parallel to the line of action of* R, that is, when either

$$\lambda = \alpha;\ \mu = \beta;\ \nu = \gamma, \ \ldots\ldots\ldots\ldots \ (11.)$$

or

$$\lambda = -\alpha;\ \mu = -\beta;\ \nu = -\gamma; \ \ldots\ldots\ldots \ (12.)$$

there is no single resultant; and the system of forces is equivalent to the force R and the couple M, being incapable of being farther simplified.

CASE V.—*When the axis of* M *is oblique to the direction of* R, making with it the angle given by the equation

$$\cos \theta = \cos \lambda \cos \alpha + \cos \mu \cos \beta + \cos \nu \cos \gamma, \ldots \ (13.)$$

the couple M is to be resolved into two rectangular components, viz:—

M $\sin \theta$ round an axis perpendicular to R, and in the plane containing the direction of R and of the axis of M;
M $\cos \theta$ round an axis parallel to R. (14.)

The force R and the couple M $\sin \theta$ are equivalent, as in Case II., to a single force equal and parallel to R, whose line of action

is in a plane perpendicular to that containing R and the axis of M, and whose perpendicular distance from O is

$$L = \frac{M \sin \theta}{R}. \qquad (15)$$

The couple M cos θ, whose axis is parallel to the line of action of R, is incapable of further combination.

Hence it appears finally, that every system of forces which is not self balanced, is equivalent either, (A); to a single force, as in Cases I. and II. (B), to a couple, as in Case III. (C); to a force, combined with a couple whose axis is parallel to the line of action of the force, as in Cases IV. and V. This can occur with inclined forces only; for the resultant of any number of parallel forces is either a single force or a couple.

101. **Parallel Projections or Transformations in Statics.** (*A. M.*, 61 to 66.)—If two figures be so related, that for each point in one there is a corresponding point in the other, and that to each pair of equal and parallel lines in the one there corresponds a pair of equal and parallel lines in the other, those figures are said to be PARALLEL PROJECTIONS of each other.

The relation between such a pair of figures is expressed algebraically as follows.—Let any figure be referred to axes of co-ordinates, whether rectangular or oblique; let x, y, z, denote the co-ordinates of any point in it, which may be denoted by A let a second figure be constructed from a second set of axes of co-ordinates, either agreeing with, or differing from, the first set as to rectangularity or obliquity, let x', y', z', be the co ordinates in the second figure, of the point A′ which corresponds to any point A in the first figure, and let those co ordinates be so related to the co ordinates of A, that for each pair of corresponding points, A, A′, in the two figures, the three pairs of corresponding co-ordinates shall bear to each other three constant ratios, such as

$$\frac{x'}{x} = a; \quad \frac{y'}{y} = b; \quad \frac{z'}{z} = c,$$

then are those two figures parallel projections of each other.

For example, all circles and ellipses are parallel projections of each other; so are all spheres, spheroids, and ellipsoids; so are all triangles; so are all triangular pyramids; so are all cylinders; so are all cones.

The following are the geometrical properties of parallel projections which are of most importance in statics:—

I. A parallel projection of a system of three points, lying in one straight line and dividing it in a given proportion, is also a

system of three points, lying in one straight line and dividing it in the same proportion.

II. A parallel projection of a system of parallel lines whose lengths bear given ratios to each other, is also a system of parallel lines whose lengths bear the same ratios to each other.

III. A parallel projection of a closed polygon is a closed polygon.

IV. A parallel projection of a parallelogram is a parallelogram.

V. A parallel projection of a parallelopiped is a parallelopiped.

VI. A parallel projection of a pair of parallel plane surfaces, whose areas are in a given ratio, is also a pair of parallel plane surfaces, whose areas are in the same ratio.

VII. A parallel projection of a pair of volumes having a given ratio, is a pair of volumes having the same ratio.

The following are the mechanical properties of parallel projections in connection with the principles set forth in this section:—

VIII. If two systems of points be parallel projections of each other; and if to each of those systems there be applied a system of parallel forces bearing to each other the same system of ratios, then the *centres of parallel forces* for those two systems of points will be parallel projections of each other, mutually related in the same manner with the other pairs of corresponding points in the two systems.

IX. If a *balanced system of forces* acting through any system of points be represented by a system of lines, then will any parallel projection of that system of lines represent a balanced system of forces; and if any two systems of forces be represented by lines which are parallel projections of each other, the lines, or sets of lines, representing their *resultants*, are corresponding parallel projections of each other,—it being observed that *couples* are to be represented by pairs of lines, as pairs of opposite forces, or by areas, and not by single lines along their axes.

SECTION III.—*Of Distributed Forces.*

102. **Distributed Forces in General.** (*A. M.*, 67, 68.)—In Article 90, p. 133, it has already been explained, that the action of every real force is distributed throughout some volume, or over some surface. It is always possible, however, to find either a *single resultant*, or a *resultant couple*, or a *combination of a single force with a couple*, to which a given distributed force is equivalent, so far as it affects the equilibrium of the body, or part of a body, to which it is applied.

In the application of Mechanics to Structures, the only force distributed throughout the volume of a body which it is necessary to consider, is its *weight*, or attraction towards the earth; and the

bodies considered are in every instance so small as compared with the earth, that this attraction may, without appreciable error, be held to act in parallel directions at each point in each body. Moreover, the forces distributed over surfaces are either parallel at each point of their surfaces of application, or capable of being resolved into sets of parallel forces; hence, *parallel distributed forces* have alone to be considered; and every such force is statically equivalent either to a single resultant, or to a resultant couple.

The *Intensity of a Distributed Force* is the ratio which the magnitude of that force, expressed in units of weight, bears to the space over which it is distributed, expressed in units of volume, or in units of surface, as the case may be. An *unit of Intensity* is an unit of force distributed over an unit of volume or of surface, as the case may be; so that there are two kinds of units of intensity. For example, *one pound per cubic foot* is an unit of intensity for a force distributed throughout a volume, such as weight; and *one pound per square foot* is an unit of intensity for a force distributed over a surface, such as pressure or friction.

103. **Weight—Specific Gravity.** (*A. M.*, 69.)—The *intensity of the weight* of a body is expressed either by stating how many units of weight are contained in an unit of volume (for example, pounds avoirdupois in a cubic foot, or in a cubic inch), or by stating the ratio which the weight of a given volume of the body bears to the weight of the same volume of a standard substance (pure water) under a standard pressure (the average atmospheric pressure of 14·7 lbs. on the square inch) and at a standard temperature (which in Britain is 62° Fahrenheit, and in France, the temperature at which water is most dense, or 39°·1 Fahr. = 3°·945 Cent). The last-mentioned ratio is called the "*Specific Gravity*" of the body. For the weight of a cubic foot, there is no single term in English: it might perhaps be called "HEAVINESS;" that being a word which at present is not appropriated to any scientific purpose. According to the French system of measures, there is no need for this distinction; because, as a litre (a cubic décimètre) of pure water at its maximum density weighs a kilogramme, the weight of a cubic décimètre of any substance in kilogrammes is its specific gravity, that of pure water being unity.

The weight of a cubic foot of pure water at 39°·1 Fahr. is

62·425 lbs. avoirdupois.

In rising from 39°·1 to 62° Fahr., pure water expands in the ratio of 1·001118 to 1, and has its density diminished in the ratio of ·998883 to 1,* hence the weight of a cubic foot of pure water at 62° Fahr. is

* See Professor Miller's paper "On the Standard Pound," *Phil. Trans.*, 1856, Part I.

$$62{\cdot}425 \times {\cdot}998883 = 62{\cdot}355 \text{ lbs. avoirdupois;}$$

and for any other substance we have,

$$\left.\begin{array}{l}\text{Heaviness in lbs. avoirdupois per cubic foot} = \text{Specific} \\ \quad \text{Gravity} \times 62{\cdot}355. \ldots\ldots\ldots\ldots\ldots\ldots\end{array}\right\} \ldots(1.)$$

In a table at the end of this volume are given the specific gravity and heaviness of such material as most commonly occur in structures. So far as that and similar tables relate to solid materials, they are approximate only; for the specific gravity of the same solid substance varies not only in different specimens, but frequently even in different parts of the same specimen; still the approximate values are sufficiently near the truth for practical purposes in the art of construction.

104. (*A. M.*, 70 to 85.) - The **Centre of Gravity** of a body, or of a system of bodies, is the point always traversed by the resultant of the weight of the body or system of bodies,— in other words, the *centre of parallel forces* for the weight of the body or system of bodies. (See Article 98.)

To *support* a body, that is, to balance its weight, the resultant of the supporting force must act through the centre of gravity.

When the centre of gravity of a *geometrical figure* is spoken of, it is to be understood to mean the point where the centre of gravity would be, if the figure were filled with a substance of uniform heaviness. The following are the most useful of the processes for finding centres of gravity.

I. If a body is *homogeneous*, or of equal specific gravity throughout, and so far *symmetrical* as to have a *centre of figure*; that is, a point within the body, which bisects every diameter of the body drawn through it, that point is also the centre of gravity of the body.

Amongst the bodies which answer this description are, the sphere, the ellipsoid, the circular cylinder, the elliptic cylinder, prisms whose bases have centres of figure, and parallelopipeds, whether right or oblique.

II. The *common centre of gravity of a set of bodies* whose several centres of gravity are known, is the *centre of parallel forces* for the weights of the several bodies, each considered as acting through its centre of gravity. (See Article 98, p. 144.)

III. If a homogeneous body be of a figure which is *symmetrical* on either side of a given plane, the centre of gravity is in that plane. If two or more such *planes of symmetry* intersect in one line, or *axis of symmetry*, the centre of gravity is in that axis. If three or more planes of symmetry intersect each other in a point, that point is the centre of gravity.

IV. To find the centre of gravity of a *homogeneous body of any figure*, assume three rectangular co-ordinate planes in any convenient position, as in fig. 67, p. 145.

To find the distance of the centre of gravity of the body from one of those planes (for example, that of Y Z), conceive the body to be divided into indefinitely thin plane layers parallel to that plane. Let s denote the area of any one of those layers, and dx its thickness, so that $s\,dx$ is the volume of the layer, and

$$V = \int s\,dx,$$

the volume of the whole body, being the sum of the volumes of the layers. Let x be the perpendicular distance of the centre of the layer $s\,dx$ from the plane of Y Z. Then the perpendicular distance x_o of the centre of gravity of the body from that plane is given by the equation

$$x_o = \frac{\int x\,s\,dx}{V} \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (1.)$$

Find, by a similar process, the distances y_o, z_o, of the centre of gravity from the other two co-ordinate planes, and its position will be completely determined.

If the centre of gravity is previously known to be in a particular plane, it is sufficient to find by the above process its distances from *two* planes perpendicular to that plane and to each other.

If the centre of gravity is previously known to be in a particular line, it is sufficient to find its distance from *one* plane, perpendicular to that line.

V. *If the specific gravity of the body varies*, let w be the mean heaviness of the layer $s\,dx$, so that

$$W = \int w\,s\,dx,$$

is the weight of the body. Then

$$x_o = \frac{\int x\,w\,s\,dx}{W} \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (2.)$$

VI. *Centre of Gravity found by Addition.*—When the figure of a body consists of parts, whose respective centres of gravity are known, the centre of gravity of the whole is to be found as in Case II.

VII. *Centre of Gravity found by Subtraction.*—When the figure of a homogeneous body, whose centre of gravity is sought, can be made by taking away a figure whose centre of gravity is known from a larger figure whose centre of gravity is known also, the following method may be used.

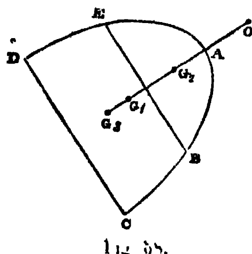


Fig. 68.

Let A C D be the larger figure, G_1 its known centre of gravity, W_1 its weight. Let A B E be the smaller figure, whose centre of gravity G_2 is known, W_2 its weight. Let E B C D be the figure whose centre of gravity G_3 is sought, made by taking away A B E from from A C D, so that its weight is

$$W_3 = W_1 - W_2.$$

Join G_1 G_2; G_3 will be in the prolongation of that straight line beyond G_1. In the same straight line produced, take any point O as origin of co-ordinates. Make $\overline{O G_1} = x_1$; $\overline{O G_2} = x_2$, $\overline{O G_3}$ (the unknown quantity) $= x_3$.

Then

$$x_3 = \frac{x_1 W_1 - x_2 W_2}{W_1 - W_2}. \quad \ldots\ldots\ldots\ldots\ldots\ldots (3.)$$

VIII. *Centre of Gravity Altered by Transposition.*—In fig. 69, let A B C D be a body of the weight W_0, whose centre of gravity G_0 is known. Let the figure of this body be altered, by transposing a part whose weight is W_1, from the position E C F to the position F D H, so that the new figure of the body is A B H E. Let G_1 be the original, and G_2 the new position of the centre of gravity of the transposed part. Then the centre of gravity of the whole body will be shifted to G_3, in a direction G_0 G_3 parallel to G_2 G_1, and through a distance given by the formula

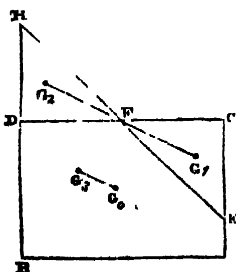


Fig. 69.

$$G_0 G_3 = \overline{G_1 G_2} \frac{W_1}{W_0}. \quad \ldots\ldots\ldots\ldots\ldots\ldots (4.)$$

IX. *Centre of Gravity found by Projection or Transformation.*—If the figures of two homogeneous bodies are parallel projections of each other, the centres of gravity of those two bodies are corresponding points in those parallel projections.

To express this symbolically,—as in Article 101, let x, y, z, be the co-ordinates, rectangular or oblique, of any point in the figure of the first body; x', y', z', those of the corresponding point in the second body; x_0, y_0, z_0, the co-ordinates of the centre of gravity of the first body: x'_0, y'_0, z'_0, those of the centre of gravity of the second body; then

$$x'_0 \quad \ldots \quad \frac{y'}{y}, \quad \frac{z'_0}{z} \qquad (5.)$$

This theorem facilitates much the finding of the centres of gravity of figures which are parallel projections of more simple or more symmetrical figures.

For example, let it be supposed that a formula is known (which will be given in p. 157) for finding the centre of gravity of a sector of a circular disc, and let it be required to find the centre of gravity of a sector of an elliptic disc. In fig. 70, let A B′ A B′ be the ellipse, A O A $= 2\,a$, and B′ O B′ $= 2\,b$, its axes, and C′ O D′ the sector whose centre of gravity is required. About the centre of the ellipse, O, describe the circle, A B A B, whose radius is the semi-axis major a. Through C′ and D′ respectively draw E C′ C and F D′ D, parallel to O B, and cutting the circle in C and D respectively; the circular sector C O D is the parallel projection of the elliptic sector C′ O D′. Let G be the centre of gravity of the sector of the circular disc, its co-ordinates being

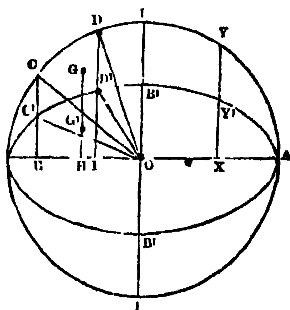


Fig. 70.

$$\mathrm{O\,H} = x_0; \ \overline{\mathrm{H\,G}} = y_0.$$

Then the co-ordinates of the centre of gravity G′ of the sector of the elliptic disc are

$$\left.\begin{aligned} \overline{\mathrm{O}}\,\mathrm{H} &= x'_0 = x_0; \\ \overline{\mathrm{H}\,\mathrm{G}'} &= y'_0 = \frac{b\,y_0}{a} \end{aligned}\right\} \ldots\ldots\ldots\ldots\ldots \qquad (6.)$$

X. *Centre of Gravity found Experimentally.*—The centre of gravity of a body of moderate size may be found approximately by experiment, by hanging it up successively by a single cord in two

different positions, and finding the single point in the body which in both positions is intersected by the axis of the cord.

105. **Examples of Weights and Centres of Gravity.** (*A. M.*, 83.)—The following examples consist of formulæ for the weight, and the position of the centre of gravity, of homogeneous bodies of those forms which most commonly occur in practice. In each case w denotes the heaviness of the body, W, its weight, and x_o, &c., the co-ordinates of its centre of gravity, which in the diagrams is marked G, the origin of co-ordinates being marked O.

A.—Prisms and Cylinders with Parallel Bases.

The word *cylinder* is here to be taken in its most general meaning, as comprehending all solids traced by the motion of a plane curvilinear figure parallel to itself.

The examples here given apply to flat plates of uniform thickness.

In the formulæ for weights, the length or thickness is supposed to be *unity*.

The centre of gravity, in each case, is at the middle of the length (or thickness); and the formulæ give its situation in the plane figure which represents the cross section of the prism or cylinder, and which is specified at the commencement of each example.

I. *Triangle.*—(Fig. 71.) O, any angle. Bisect opposite side B C in D. Join O D.

$$x = OG = \frac{2}{3} OD.$$

$$W = w \cdot \frac{OD \cdot BC \cdot \sin \angle ODC}{2}.$$

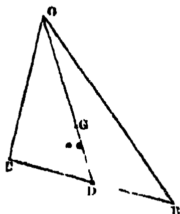


Fig. 71.

II. *Polygon.*—Divide it into triangles; find the centre of gravity of each; then find their common centre of gravity as in Article 104, Case II., p. 152.

III. *Trapezoid.*—(Fig. 72.) A B ∥ C E.

Greatest breadth, A B = B.
Least " C E = b.
Bisect A B in O, C E in D; join O D.

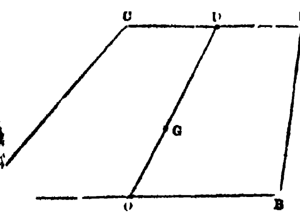


Fig. 72.

$$x_o = \overline{OG} = \frac{\overline{OD}}{2}\left(1 - \frac{1}{3} \cdot \frac{B-b}{B+b}\right)$$

$$W = w \cdot \overline{OD} \cdot \frac{B+b}{2} \cdot \sin \angle DOB.$$

IV. *Trapezoid.*—(Second solution.)—(Fig. 73.) O, point where inclined sides meet. Let $\overline{OF} = x_1$, $\overline{OD} = x_2$, $\overline{OG} = x_o$.

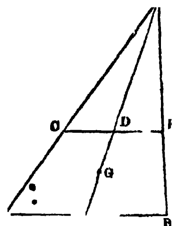


Fig. 73.

$$x_o = \frac{2}{3} \cdot \frac{x_1^3 - x_2^3}{x_1^2 - x_2^2}.$$

$$W = w \cdot \frac{x_1^2 - x_2^2}{2} \cdot \sin^2 \angle OFB.$$

$$(\text{cotan} \angle OAB + \text{cotan} \angle OBA).$$

V. *Parabolic Half-Segment.*—(O A B, fig. 74.) O, vertex of diameter O X; $\overline{OA} = x_1$; A B $= y_1$, ordinate $\|$ tangent O C Y.

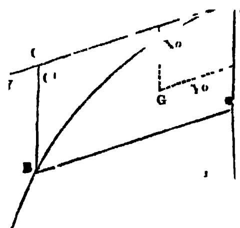


Fig. 74.

$$x_o = \frac{3}{5} x_1; \; y_o = \frac{3}{8} y_1.$$

$$W = \frac{2}{3} w \cdot x_1 y_1 \cdot \sin \angle XOY.$$

VI. *Parabolic Spandril.*—(O B C, fig. 74.) G′, centre of gravity,

$$x_o = \frac{3}{10} x_1; \; y_o = \frac{3}{4} y_1; \; W = \frac{1}{3} w \; x_1 y_1 \; \sin \angle XOY.$$

VII. *Circular Sector.*—(O A C, fig. 75.) Let O X bisect the angle A O C; O Y ⊥ O X.

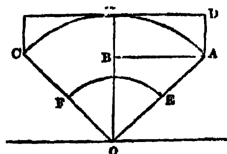


Fig. 75.

Radius $\overline{OA} = r$

Half-arc, to radius unity, $\frac{AC}{2\,AO} = \theta$.

$$x_o = \frac{2}{3} r \frac{\sin \theta}{\theta}; \; y_o = 0.$$

$$W = w r^2 \theta$$

VIII. *Circular Half-Segment.*—(A B X, Fig. 75.)

$$x_0 = \frac{2}{3} r \cdot \frac{\sin^3 \theta}{\theta - \sin \theta \cos \theta}; \quad y_0 = r \cdot \frac{4 \sin^2 \frac{\theta}{2} - \sin^2 \theta \cos \theta}{3 (\theta - \cos \theta \sin \theta)}.$$

$$W = \frac{1}{2} w r^2 (\theta - \cos \theta \sin \theta).$$

IX. *Circular Spandril.*—(A D X, Fig. 75.)

$$x_0 = \frac{1}{3} r \cdot \frac{\sin^3 \theta}{2 \sin \theta - \sin \theta \cos \theta - \theta};$$

$$y_0 = \frac{1}{3} r \cdot \frac{3 \sin^2 \theta - 2 \sin^2 \theta \cos \theta - 4 \sin^2 \frac{\theta}{2}}{2 \sin \theta - \sin \theta \cos \theta - \theta}.$$

$$W = w r^2 \cdot \left(\sin \theta - \frac{1}{2} \sin \theta \cos \theta - \frac{\theta}{2} \right).$$

X. *Sector of Ring.*—(A C F E, Fig. 75.) $\overline{OA} = r$; $\overline{OE} = r'$.

$$x_0 = \frac{2}{3} \cdot \frac{r^3 - r'^3}{r^2 - r'^2} \cdot \frac{\sin \theta}{\theta}; \quad y_0 = 0.$$

$$W = w (r^2 - r'^2) \theta.$$

XI. *Elliptic Sector, Half-Segment, or Spandril.*—Centre of gravity to be found by projection from that of corresponding circular figure, as in Article 104, Case IX., p. 154.

B.—Wedges.

XII. *General Formulæ for Wedges.*—(Fig. 76.) All wedges may be divided into parts such as the figure here represented. O A Y, O X Y, planes meeting in the edge O Y; A X Y, cylindrical (or prismatic) surface perpendicular to the plane O X Y; O X A, plane triangle perpendicular to the edge O Y; O Z, axis perpendicular to X O Y. Let $\overline{OX} = x_1$; $\overline{XA} = z_1$. Then $z = \frac{z_1 x}{x_1}$;

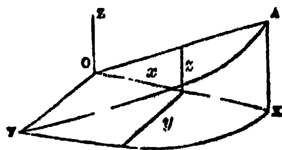


Fig. 76.

$$W = w \cdot \frac{z_1}{x_1} \int x y \cdot dx$$

$$x_0 = \frac{\int x^2 y \cdot dx}{\int x y \cdot dx}; \quad y_0 = \frac{\int x y^2 \cdot dx}{2 \int x y \cdot dx}; \quad z_0 = \frac{z_1 x_0}{2 x_1}.$$

(This last equation denotes that G is in the plane which traverses O Y and bisects A X.)

In a symmetrical wedge, if O be taken at the middle of the edge, $y_0 = 0$. Such is the case in the following examples, in each of which, length of edge $= 2\,y_1$.

XIII. *Rectangular Wedge.*—(= Triangular Prism.)—(Fig. 77.)

$$W = w \cdot x_1 y_1 z_1;\ x_0 = \frac{2}{3} x_1.$$

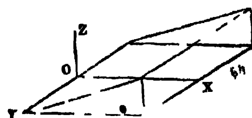


Fig. 77.

XIV. *Triangular Wedge* —(= Triangular Pyramid.)—(Fig. 78.)

$$W = \frac{1}{3} w \cdot x_1 y_1 z_1;\ x_0 = \frac{1}{2} x_1.$$

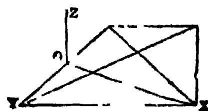


Fig. 78.

XV. *Semicircular Wedge.*—(Fig. 79.)

Radius $\overline{OX} = \overline{OY} = r$.

$$W = \frac{2}{3} w \cdot r^2 z_1;\ x_0 = \frac{3\pi}{16} r = \cdot 58905\, r.$$

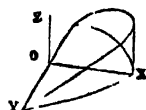


Fig 79.

XVI. *Annular, or Hollow Semicircular Wedge.*—(Fig. 80.)

External radius, r; internal r'.

$$W = \frac{2}{3} w \cdot \left(r^3 - r'^3\right) \frac{z^1}{r};\ x_0 = \frac{3\pi}{16} \cdot \frac{r^4 - r'^4}{r^3 - r'^3}.$$

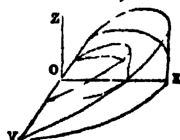


Fig. 80.

C.—Cones and Pyramids.

Let O denote the apex of the cone or pyramid, taken as the origin, and X the centre of gravity of a supposed flat plate whose middle section coincides with the base of the cone, or pyramid. The centre of gravity will lie in the axis O X.

Denote the area of the base by A, and the angle which it makes with the axis by θ.

XVII. *Complete Cone or Pyramid.*—Let the height O X $= h$;

$$x_0 = \frac{3}{4} h; \quad W = \frac{1}{3} w \cdot A h \sin \theta.$$

XVIII. *Truncated Cone or Pyramid.*—Height of portion truncated $= h'$.

$$x_0 = \frac{3}{4} \cdot \frac{h^4 - h'^4}{h^3 - h'^3}; \quad W = \frac{1}{3} w A h \cdot \left(1 - \frac{h'^3}{h^3}\right) \sin \theta.$$

D.—Portions of a Sphere.

XIX. *Zone or Ring of a Spherical Shell,* bounded by two conical surfaces having their common apex at the centre O of the sphere (fig. 81).

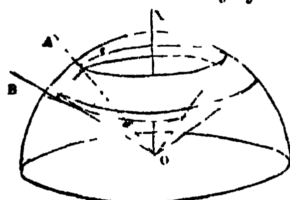


Fig. 81.

O X, axis of cones and zone.
r, external radius } of shell.
r', internal radius } of shell.
$\angle$ X O A $= \alpha$, half-angle of less } cone.
$\angle$ X O B $= \beta$, greater } cone.

$$x_0 = \frac{3}{4} \cdot \frac{r^4 - r'^4}{r^3 - r'^3} \cdot \frac{\cos \alpha + \cos \beta}{2}$$

$$W = \frac{2 \pi w}{3} \left(r^3 - r'^3\right) \cdot \left(\cos \beta - \cos \alpha\right).$$

XX. *Sector of a Hemispherical Shell.*—(C X D, fig. 82.) O Y bisects angle D O C; $\frac{1}{2}$ D O C $= \theta$.

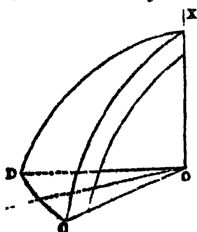


Fig. 82.

$$x_0 = \frac{3}{8} \cdot \frac{r^4 - r'^4}{r^3 - r'^3}; \quad y_0 = \frac{3 \pi}{16} \cdot \frac{r^4 - r'^4}{r^3 - r'^3} \cdot \frac{\sin \theta}{\theta}.$$

$$W = \frac{2 \theta w}{3} \left(r^3 - r'^3\right).$$

106. **Stress—its Intensity, Resultant, Centre, and Moment.** (*A. M.*, 86 to 89.)—The word STRESS has been adopted as a general term to comprehend various forces which are exerted between contiguous bodies, or parts of bodies, and which are distributed over the surface of contact of the masses between which they act.

The INTENSITY of a stress is its amount in units of weight, divided by the extent of the surface over which it acts, in units of area.

The following table gives a comparison of various units in which the intensity of stress is expressed:—

	Pounds on the square foot.	Pounds on the square inch.
One pound on the square inch,.....	144	1
One pound on the square foot,.....	1	$\frac{1}{144}$
One inch of mercury (that is, weight of a column of mercury at 32° Fahr., one inch high), ...	70·73	0·4912
One foot of water (at 39°·1 Fahr.),	62·425	0·4335
One inch of water (at 39°·1 Fahr.),	5·2021	0·036125
One foot of water (at 62° Fahr.),...	62·355	0·43302
One inch of water (at 62° Fahr.),...	5·19625	0·036085
One atmosphere, of 29·922 inches of mercury, or 760 millimètres,	2116·4	14·7
One foot of air, at 32° Fahr., and under the pressure of one atmosphere, ...	0·080728	0·0005606
One kilogramme on the square mètre, ...	0·20481	0·00142228
One kilogramme on the square millimètre,......	204810	1422·28
One millimètre of mercury,.........	2·7847	0·01934

The various kinds of stress may be thus classed:—

I. *Thrust*, or *Pressure*, is the force which acts between two contiguous bodies, or parts of a body, when each pushes the other from itself.

II. *Pull*, or *Tension*, is the force which acts between two contiguous bodies, or parts of a body, when each draws the other towards itself.

Pressure and tension may be either *normal* or *oblique*, relatively to the surface at which they act.

III. *Shear*, or *Tangential Stress*, is the force which acts between two contiguous bodies, or parts of a body, when each draws the other sideways, in a direction parallel to their surface of contact.

In expressing a Thrust and a Pull in parallel directions algebrai-

cally, if one is treated as positive, the other must be treated as negative. The choice of the positive or negative sign for either is a matter of convenience.

The word "*Pressure*," although, strictly speaking, equivalent to "*thrust*," is sometimes applied to *stress* in general; and when this is the case, it is to be understood that thrust is treated as positive.

The following are the processes for finding the *magnitude of the resultant* of a stress distributed over a plane surface, and the *centre of stress*; that is, the point where the line of action of that resultant cuts the plane surface:—

I. *If the stress is of uniform intensity*, the magnitude of its resultant is the product of that intensity and the area of the surface; and the centre of stress is at the centre of gravity of the surface. Or in symbols, let S be the area of the surface, p the intensity of the stress, P its resultant, then—

$$P = pS. \qquad (1.)$$

II. *If the stress is of varying intensity, but of one sign;* that is, all tension, or all pressure, or all shear in one direction.

In fig. 83, let A A be the given plane surface at which the stress acts; O X, O Y, two rectangular axes of co-ordinates in its plane; O Z, a third axis perpendicular to that plane. Conceive a solid to exist, bounded at one end by the given plane surface A A, laterally by a cylindrical or prismatic surface generated by the motion of a straight line parallel to O Z round the outline of A A, and at the other end by a surface B B, of such a figure, that its ordinate z at any point shall be proportional to the intensity of the stress at the point a of the surface A A from which that ordinate proceeds, as shown by the equation

Fig. 83.

$$z = \frac{p}{w} \qquad (2.)$$

Conceive the surface A A to be divided into an indefinite number of small rectangular areas, each denoted by $dx\,dy$, and so small that the stress on each is sensibly uniform; the entire area being

$$S = \iint dx\,dy.$$

The volume of the ideal solid will be

$$V = \iint z \cdot dx\,dy \ldots\ldots\ldots\ldots\ldots\ldots (3.)$$

So that if it be conceived to consist of a material whose heaviness

is $w = \frac{p}{z}$, the amount of the stress will be equal to the weight of the solid; that is to say,

$$P = \iint p\,dx\,dy = w\,V \ldots\ldots\ldots\ldots\ldots\ldots\ldots (4.)$$

The *centre of stress* is the point on the surface A A perpendicularly opposite the centre of gravity of the ideal solid.

The simplest, and at the same time the commonest case of this kind is where the stress is *uniformly-varying;* that is, where its intensity at a given point is simply proportional to the perpendicular distance of that point from a given straight line in the plane of the surface A A. The ideal solid is now either a wedge, or a figure that can be made by adding and subtracting wedges; so that the resultant and centre of stress are to be found by the methods of Article 105, Cases XII. to XVI., and Article 104, Cases II. and VII. To express this symbolically, take the straight line in question for the axis O Y; conceive the surface to be divided into bands by lines parallel to O Y; let y denote the length of one of these bands, and dx its breadth, so that $y\,dx$ is its area, and $S = \int y\,dx$ the area of the whole surface. Let x be the perpendicular distance of the centre of a band from the *line of no stress* O Y, and let the intensity of the stress there be

$$p = a\,x; \ldots\ldots\ldots\ldots\ldots\ldots\ldots (5.)$$

a being a constant co-efficient, then the amount or resultant of the stress is

$$P = \int p\,y\,dx = a \int x\,y\,dx; \ldots\ldots\ldots\ldots\ldots (6.)$$

and the perpendicular distance of the centre of stress from O Y is

$$x_0 = \frac{\int p\,x\,y\,dx}{\int p\,y\,dx} = \frac{a \int x^2\,y\,dx}{P} \ldots\ldots\ldots\ldots\ldots (7.)$$

Examples of this case will be given in treating of the pressure of water and of earth, and the stability of masonry.

III. *When the stress is of contrary signs;* for example, pressure at one part of the surface and tension at another, the resultants and centres of stress of the pressure and tension are to be found separately. Those partial resultants are then to be treated as a pair of parallel forces acting through the two respective centres of stress; their final resultant will be equal to their difference, if any, acting through a point found as in Article 97, Case IV., p. 141.

If the total pressure and total tension are equal to each other, they have no single resultant and no single centre of stress: their resultant being a couple, whose moment is equal to the total stress

of either kind multiplied by the perpendicular distance between the resultant of the pressure and the resultant of the tension. Examples of this case will be given in treating of the strength of beams.

107. **Pressure and Balance of Fluids—Principles of Hydrostatics.**—*Fluid* is a term opposed to *solid*, and comprehending the liquid and gaseous conditions of bodies. The property common to the liquid and the gaseous conditions is that of *not tending to preserve a definite shape*, and the possession of this property by a body in perfection throughout all its parts, constitutes that body a *perfect fluid.*

A necessary consequence of that property is the following principle, which is the foundation of the whole science of hydrostatics:—

I. *In a perfect fluid, when still, the pressure exerted at a given point is normal to the surface on which it acts, and of equal intensity for all positions of that surface.*

The following are some of the most useful consequences of that principle:—

II. *A surface of equal pressure in a still fluid mass is everywhere perpendicular to the direction of gravity;* that is, horizontal throughout. In other words, the pressure at all points at the same level is of equal intensity.

III. *The intensity of the pressure at the lower of two points in a still fluid mass is greater than the intensity at the higher point, by an amount equal to the weight of a vertical column of the fluid whose height is the difference of elevation of the points, and base an unit of area.*

To express this symbolically, let p_0 denote the intensity of the pressure at the higher of two points in a fluid mass, and p_1 the intensity at a point whose vertical depth below the former point is x. Let w be the *mean heaviness* of the layer of fluid between those two points; then

$$p_1 - p_0 + wx. \quad\text{.........................}\quad (1)$$

In a gas, such as air, w varies, being nearly proportional to p; but in a liquid, such as water, the variations of w are too small to be considered in practical cases.

For example, let the upper of the two points be the surface of a mass of water where it is exposed to the air; then p_0 is the atmospheric pressure; let the depth x of the second point below the surface be given in feet, and let the temperature be 39°·1; then

$$p_1 \text{ in lbs. on the square foot} = p_0 + 62{\cdot}425\, x. \quad\text{.........}(2.)$$

In many questions relating to engineering, the pressure of the atmosphere may be left out of consideration, as it acts with sensibly equal intensity on all sides of the bodies exposed to it, and so

balances its own action. The pressures calculated, in such cases, is the *excess* of the pressure of the water above the atmospheric pressure, which may be thus expressed,—

$$p' = p_1 - p_0 = 62{\cdot}4\ x \text{ nearly.} \quad \ldots\ldots\ldots\ldots(3.)$$

IV. The pressure of a liquid on a *floating* or *immersed body*, is equal to the weight of the volume of fluid displaced by that body; and the resultant of that pressure acts vertically upwards through the centre of gravity of that volume; which centre of gravity is called the "*centre of buoyancy*."

V. The pressure of a liquid against *a plane surface immersed in it* is perpendicular to that surface in direction: its magnitude is equal to the weight of a volume of the liquid, found by multiplying the area of the surface by the depth to which its centre of gravity is immersed.

VI. The *centre of pressure* on such a surface, if the surface is horizontal, coincides with its centre of gravity; if the surface is vertical or sloping, the centre of pressure is always below the centre of gravity of the surface, and is found by considering that the pressure is an *uniformly-varying* stress, whose intensity at a given point varies as the distance of that point from the line where the given plane surface (produced if necessary) intersects the upper surface of the liquid.

To express the last two principles by symbols in the case in which the pressed surface is vertical or sloping, let the line where the plane of that surface cuts the upper surface of the liquid be taken as the axis O Y. Let θ denote the angle of inclination of the pressed surface to the horizon. Conceive that surface to be divided by parallel horizontal lines into an indefinite number of narrow bands. Let y be the length of any one of those bands, dx its breadth, x the distance of its centre from O Y; then $y\,dx$ is its area, $x \sin\theta$ the depth at which it is immersed; and if w be the weight of unity of volume of the fluid, the intensity of the pressure on that band is

$$p = w\,x \sin\theta. \quad \ldots\ldots\ldots\ldots (4.)$$

The whole area of the pressed surface, being the sum of the areas of all the bands, is $S = \int y\,dx$; the whole pressure upon it is

$$P = \int p\,y\,dx = w \sin\theta \int x\,y\,dx; \quad \ldots\ldots\ldots\ldots(5.)$$

The mean intensity of the pressure is

$$\frac{P}{S} = \frac{\int p\,y\,dx}{\int y\,dx} = w \sin\theta \cdot \frac{\int x\,y\,dx}{\int y\,dx}; \quad \ldots\ldots\ldots\ldots(6.)$$

and the distance of the centre of pressure from O Y is

$$x_0 = \frac{\int x p y \, dx}{P} = \frac{\int x^2 y \, dx}{\int x y \, dx}. \qquad (7.)$$

For example, let the sloping pressed surface be rectangular, like a sluice, or the back of a reservoir-wall; and, in the first instance, let it extend from the surface of a mass of water down to a distance x_1, measured along the slope, so that its lower edge is immersed to the depth $x_1 \sin \theta$. Then its centre of gravity is immersed to the depth $x_1 \sin \theta \div 2$, and the mean intensity of the pressure in lbs. on the square foot, is

$$\frac{P}{S} = \frac{62 \cdot 4 \, x_1 \sin \theta}{2}. \qquad (8.)$$

The breadth y is constant; so that the area of the surface is $S = x_1 y$; and the total pressure is

$$P = \frac{62 \cdot 4 \, x_1^2 \, y \sin \theta}{2} \qquad (9.)$$

The distance of the centre of pressure from the upper edge is

$$x_0 = \frac{2}{3} \cdot x_1 \qquad (10.)$$

Next, let the upper edge, instead of being at the surface of the water, be at the distance x_2 from it, so as to be immersed to the depth $x_2 \sin \theta$. Then the centre of gravity of the pressed surface is immersed to the depth $(x_1 + x_2) \sin \theta \div 2$, and the mean intensity of the pressure upon it, in lbs. on the square foot, is

$$\frac{P}{S} = \frac{62 \cdot 4 \, (x_1 + x_2) \sin \theta}{2}; \qquad (11.)$$

the area of the surface is $(x_1 - x_2) \, y$, and the total pressure on it

$$P = \frac{62 \cdot 4 \, (x_1^2 - x_2^2) \, y \sin \theta}{2}. \qquad (12.)$$

The distance of the centre of pressure from the line O Y is

$$x_0 = \frac{2}{3} \cdot \frac{x_1^3 - x_2^3}{x_1^2 - x_2^2}. \qquad (13.)$$

108. **Compound Internal Stress of Solids.** (*A. M.*, 96 to 113.)—If a body be conceived to be divided into two parts by an ideal plane traversing it in any direction, the force exerted between those two parts at the plane of division is an *internal stress*.

According to the principles stated in the preceding article, the internal stress at a given point in a fluid is normal and of equal intensity for all positions of the ideal plane of division. In a solid body, on the other hand, the stress may be either normal, oblique, or shearing; and it may vary in direction and intensity, as the position of the ideal plane of division varies.

If the direction and intensity of the stress at a given point in a solid mass are given, for three positions of the plane of division, they can be found for any position whatsoever. It is unnecessary in the present treatise to give the methods of solving this problem in all its generality. Certain particular cases only will be given, which are useful in the theories of the stability of earth and of the strength of materials.

I. *Conjugate Stresses—Principal Stresses.*—If two planes traverse a point in a body, and the direction of the stress on the first plane is parallel to the second plane, then the direction of the stress on the second plane is parallel to the first plane. Such a pair of stresses are said to be *conjugate;* and if they are both normal to their planes of application (and consequently perpendicular to each other) they are called *principal stresses.* Three conjugate stresses, or three principal stresses, may act through one point; but in the present treatise it is sufficient to consider two.

Fig. 84 represents a pair of conjugate oblique tensions acting in the directions X X and Y Y through a prismatic particle A B C D.

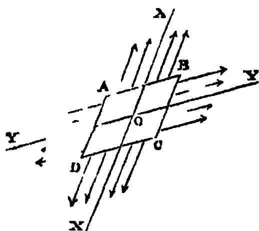


Fig. 84.

The rectangular directions in which principal stresses—that is, direct pulls and thrusts—act, through a given point in a solid, are called *axes of stress.*

In a fluid, the stress at a given point being of equal intensity in all directions, every direction has the property of an axis of stress. A solid *may* be in the same condition with a fluid as to stress; but it may also have the principal stresses at a given point of different intensities. (In a mass of loose grains, the ratio of those intensities has a limit depending on friction, as will afterwards be more fully explained in treating of the stability of earth:—in a firm continuous solid, the principal stresses at a point may bear any ratio to each other, and may be either of the same or of opposite kinds.

II. *The Shearing Stress,* on two planes traversing a point in a solid at right angles to each other, is of equal intensity.

III. *A Pair of Equal and Opposite Principal Stresses;* that is, a pull and a thrust of equal intensity acting through a particle of a

solid in directions at right angles to each other, are equivalent to a pair of shearing stresses of the same intensity on a pair of planes at right angles to each other, and making angles of 45° with the first pair of planes.

IV. *Combination of any Two Principal Stresses.*

PROBLEM.—A pair of principal stresses of any intensities, and of the same or opposite kinds, being given, it is required to find the direction and intensity of the stress on a plane in any position at right angles to the plane parallel to which the two principal stresses act.

Let O X and O Y (figs. 85 and 86) be the directions of the two principal stresses; O X being the direction of the greater stress.

Let p_1 be the intensity of the greater stress;

and p_2 that of the less.

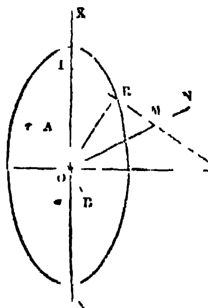


Fig 85.

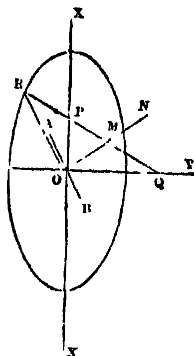


Fig. 86.

The kind of stress to which each of these belongs, pull or thrust, is to be distinguished by means of the algebraical signs. If a pull is considered as positive, a thrust is to be considered as negative, and *vice versâ*. It is in general convenient to consider that kind of stress as positive to which the greater principal stress belongs. Fig. 85 represents the case in which p_1 and p_2 are of the same kind; fig. 86 the case in which they are of opposite kinds In all the following equations, the sign of p_2 is held to be *implied* in that symbol; that is to say, when p_2 is of the contrary kind to p_1, the sign applied to its arithmetical value, in computing by means of the equations, is to be reversed.

Let A B be the plane on which it is required to ascertain the direction and intensity of the stress, and O N a normal to that plane, making with the axis of greatest stress the angle

$$\angle \text{X O N} = \widehat{xn}.$$

On O N take $\overline{OM} = \frac{p_1 + p_2}{2}$; this will represent a normal stress on A B of the same kind with the greater principal stress, and of an intensity which is a mean between the intensities of the two principal stresses.

Through M draw P M Q, making with the axes of stress the same angles which O N makes, but in the opposite direction; that is to say, take $\overline{MP} = \overline{MQ} = \overline{MO}$. On the line thus found set off from M towards the axis of greatest stress, $\overline{MR} = \frac{p_1 - p_2}{2}$.

Join $\overline{OR}$. Then will that line represent the direction and intensity of the stress on A B.

The algebraical expression of this solution is easily obtained by means of the formulæ of plane trigonometry, and consists of the two following equations:—

$$\text{Intensity, } \overline{OR} \text{ or } p = \sqrt{\{p_1^2 \cdot \cos^2 \hat{xn} + p_2^2 \cdot \sin^2 \hat{xn}\}} \ldots (1.)$$

$$\text{Angle of obliquity, N O R or } \hat{nr}$$

$$= \text{arc sin} \cdot \left(\sin 2\,\hat{xn} \cdot \frac{p_1 - p_2}{2p}\right) \ldots\ldots\ldots (2.)$$

This obliquity is always towards the axis of greatest stress.

In fig. 85, p_1 and p_2 are represented as being of the same kind; and $\overline{MR}$ is consequently less than $\overline{OM}$, so that $\overline{OR}$ falls on the same side of O X with O N; that is to say, $\hat{nr} < \hat{xn}$. In fig. 86, p_1 and p_2 are of opposite kinds, M R is greater than $\overline{OM}$, and O R falls on the opposite side of O X to O M, that is to say, $\hat{nr} > \hat{xn}$.

The locus of the point M is a circle of the radius $\frac{p_1 + p_2}{2}$, and that of the point R, an ellipse whose semi-axes are p_1 and p_2, and which may be called the Ellipse of Stress, because its semi-diameter in any direction represents the intensity of the stress in that direction.

V. Deviation of Principal Stresses by a Shearing Stress.

Problem.—Let p_x and p_y denote the original intensities of a pair of principal stresses acting at right angles to each other through one particle of a solid. Suppose that with these there is combined a shearing stress of the intensity q, acting in the same plane with the original pulls or thrusts; it is required to find the new intensities and new directions of the principal stresses.

To assist the conception of this problem, the original stresses

referred to are represented in fig. 87, as acting through a particle of the form of a square prism. The principal stresses, both original and new, are represented as tensions, although any or all of them might be pressures. In the formulæ annexed, tensions are considered positive, pressures negative; angles lying to the right of A A are considered as positive, to the left as negative; and a shearing stress is considered as positive or negative according as it tends to make the upper right-hand and lower left-hand corner of the square particle acute or obtuse.

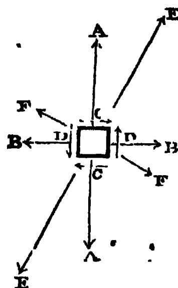


Fig. 87.

The arrows A A represent the greater original tension p_x; the angles B B, the less original tension p_y; C, C, D, D, represent the positive shear of the intensity q, as acting at the four faces of the particle. The combination of this shear with the original tensions is equivalent to a new pair of principal tensions, oblique to the original pair. The greater new principal tension, p_1, is represented by the arrows E, E; it deviates to the right of p_x through an angle which will be denoted by θ. The less new principal tension p_2 is represented by the arrows F, F; it deviates through the same angle to the right of p_y.

Then the intensities of the new principal stresses are given by the equations;

$$\left. \begin{aligned} p_1 &= \frac{p_x + p_y}{2} + \sqrt{\left\{ \frac{(p_x - p_y)^2}{4} + q^2 \right\}}; \\ p_2 &= \frac{p_x + p_y}{2} - \sqrt{\left\{ \frac{(p_x - p_y)^2}{4} + q^2 \right\}}; \end{aligned} \right\} \quad (3)$$

and the double of the angle of deviation by either of the following;

$$\tan 2\theta = \frac{2q}{p_x - p_y}; \text{ or cotan } 2\theta = \frac{p_x - p_y}{2q}. \quad \ldots\ldots (4)$$

The greatest value of θ is 45°, when $p_x = p_y$.

The new principal stresses are to be conceived as acting normally on the faces of a new square prism.

109. **Parallel Projection of Distributed Forces.**—In applying the principles of parallel projection to distributed forces, it is to be borne in mind that those principles, as stated in Article 101, are applicable to lines representing the *amounts* or *resultants* of distributed forces, and *not their intensities*. The relations amongst the intensities of a system of distributed forces, whose resultants have been obtained by the method of projection, are to be arrived at by a subsequent process of dividing each projected resultant by the projected space over which it is distributed.

110. **Friction** (*A. M.*, 189, 190, 191) is that force which acts between two bodies at their surface of contact so as to resist their sliding on each other, and which depends on the force with which the bodies are pressed together. It is a kind of shearing stress. The following law respecting the friction of solid bodies has been ascertained by experiment:—

The friction which a given pair of solid bodies, with their surfaces in a given condition, are capable of exerting, is simply proportional to the force with which they are pressed together.

If a body be acted upon by a force tending to make it slide on another, then so long as that force does not exceed the amount fixed by this law, the friction will be equal and opposite to it, and will balance it.*

There is a limit to the exactness of the above law, when the pressure becomes so intense as to crush or indent the parts of the bodies at and near their surface of contact. At and beyond that limit the friction increases more rapidly than the pressure; but that limit ought never to be attained in any structure. For some substances, especially those whose surfaces are sensibly indented by a moderate pressure, such as timber, the friction between a pair of surfaces which have remained for some time at rest relatively to each other, is somewhat greater than that between the same pair of surfaces when sliding on each other. That excess, however, of the *friction of rest* over the *friction of motion*, is instantly destroyed by a slight vibration; so that the *friction of motion* is alone to be taken into account, as contributing to the stability of a structure.

The friction between a pair of surfaces is calculated by multiplying the force with which they are directly pressed together, by a factor called the *co-efficient of friction*, which has a special value depending on the nature of the materials and the state of the surfaces. Let F denote the friction between a pair of surfaces; N, the force, in a direction perpendicular to the surfaces, with which they are pressed together; and f the co-efficient of friction; then

$$F = f N \qquad (1.)$$

The co-efficient of friction of a given pair of surfaces is the *tangent* of an angle called the *angle of repose*, being the greatest angle which an oblique pressure between the surfaces can make with a perpendicular to them, without making them slide on each other.

Let P denote the amount of an oblique pressure between two plane surfaces, inclined to their common normal at the angle of repose φ; then

$$F = f N = N \tan \varphi = P \sin \varphi = \frac{f P}{\sqrt{1 + f^2}} \qquad (2.)$$

* See Addendum, pp. 789 and 797.

The angle of repose is the steepest inclination of a plane to the *horizon, at which a block of a given substance will remain balanced on it without sliding down.*

The *intensity* of the friction between two surfaces bears the same proportion to the intensity of the pressure that the whole friction bears to the whole pressure.

The following is a table of the angle of repose φ, the co-efficient of friction $f = \tan \varphi$, and its reciprocal $1 : f$, for various materials—condensed from the tables of General Morin, and other sources, and arranged in a few comprehensive classes. The values of those constants which are given in the table have reference to the *friction of motion.*

Surfaces.			$\frac{1}{f}$
Dry masonry and brickwork, ...	31° to 35°	0·6 to 0·7	1·67 to 1·43
Masonry and brickwork with wet mortar,	25½°	0·47	2·1
Masonry and brickwork, with slightly damp mortar..........		0·74	1·35
Wood on stone,..		about 0·4	2·5
Iron on stone,................................	35° to 16⅔°	0·7 to 0·3	1·43 to 3·33
Masonry on dry clay,...............	27°	0·51	1·96
„ on moist clay,....................	18¼°	0·33	3
Earth on earth, ...	14° to 45°	0·25 to 1.0	4 to 1
„ „ dry sand, clay, and mixed earth............................	21° to 37°	0·38 to 0·75	2·63 to 1·33
„ „ damp clay,..	45°	1·0	1
„ „ wet clay,....	17°	0·31	3·23
„ „ shingle and gravel,.......	35° to 48°	0·7 to 1·11	1·43 to 0·9
Wood on wood, dry,........................	11° to 26½	·25 to ·5	4 to 2
„ „ soaped, ...	11½° to 2°	·2 to ·04	5 to 25
Metals on oak, dry, ...	26½° to 31°	·5 to ·6	2 to 1·67
„ „ wet, ...	13½° to 14½°	·24 to ·26	4·17 to 3·85
[illegible]py,	11½°	·2	5
Metals on elm, dry,	11½° to 14°	·2 to ·25	5 to 4
Bronze on lignum vitæ, constantly wet.	3° ?	·05 ?	20 ?
Hemp on oak, dry, ...	28°	·53	1·89
„ „ wet, ...	18½°	·33	3
Leather on oak,..............................	15° to 19½°	·27 to ·38	3·7 to 2·86
Leather on metals, dry, ...	29½°	·56	1·79
„ „ wet, ...	20°	·36	2·78
„ „ greasy, ...	13°	·23	4·35
„ „ oily, ...	8½°	·15	6·67
Metals on metals, dry, ...	8½° to 11½°	·15 to ·2	6·67 to 5
„ „ wet and clean, ...	16½°	·3	3·33
„ „ damp and slimy,......	8°	·14	7·14
Smooth surfaces, occasionally greased,...	4° to 4½°	·07 to ·08	14·3 to 12·5
„ „ continually greased, ...	3°	·05	20
Smoothest and best greased surfaces, ...	1¾° to 2°	·03 to ·036	33·3 to 27·6

Section IV.—*Balance and Stability of Frames, Chains, Ribs, and Blocks.*

(*A. M.*, 137 to 211.)

111. A **Frame** is a structure composed of bars, rods, links, or cords, attached together or supported by *joints*, such as occur in carpentry, in frames of metal bars, and in structures of ropes and chains, fixing the ends of two or more pieces together, but offering little or no resistance to change in the relative angular positions of those pieces. In a joint of this class, the *centre of resistance*, or point through which the resultant of the resistance to displacement of the pieces connected at the joint acts, is at or near the middle of the joint, and does not admit of any variation of position consistently with security.

The *Line of Resistance* of a frame is a line traversing the *centres of resistance* of the joints, and is in general a polygon, having its angles at these centres.

112. A **Single Bar** in a frame may act as a Tie, a Strut, or a Beam. (*A. M.*, 138 to 142)

I. A *Tie* has equal and directly opposite forces applied to its two ends, acting outwards, or *from* each other. The bar is in a state of *tension*, and the stress exerted between any two divisions of it is a *pull*, equal and opposite to the applied forces. A *rope* or *chain* will answer the purpose of a tie.

The equilibrium of a moveable tie is stable, for if its angular position be deviated, the forces applied to its ends, which originally were directly opposed, now constitute a *couple* tending to restore the tie to its original position.

II. A *Strut* has equal and directly opposite forces applied to its two ends, acting inwards, or *towards* each other. The bar is in a state of compression, and the stress exerted between any two divisions of it is a *thrust* equal and opposite to the applied forces. It is obvious that a flexible body will *not* answer the purpose of a strut.

The equilibrium of a moveable strut is unstable; for if its angular position be deviated, the forces applied to its ends, which originally were directly opposed, now constitute a couple tending to make it deviate still farther from its original position.

In order that a strut may have stability, its ends must be prevented from deviating laterally. Pieces connected with the ends of a strut for this purpose are called *stays.*

III. A *Beam* is a bar supported at two points, and loaded in a direction perpendicular or oblique to its length.

Case I.—Let the supporting pressures be parallel to each

other and to the direction of the load; and let the load act ***between*** the points of support, as in fig. 88; where P represents the resultant of the gross load, including the weight of the beam itself, L, the point where the line of action of that resultant intersects the axis of the beam; R_1, R_2, the two supporting pressures or resistances of the props parallel to, and in the same plane with P, and acting through the points S_1, S_2, in the axis of the beam.

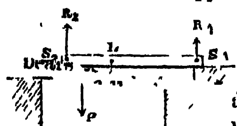


Fig. 88.

Then, according to the principle of the lever, Article 97, p. 141, each of those three forces is proportional to the distance between the lines of action of the other two; and the load is equal to the sum of the two supporting pressures; that is to say,

$$P : R_1 : R_2 :: \overline{S_1 S_2} : \overline{L S_2} : \overline{L S_1}; \qquad (1.)$$

$$\text{and } P = R_1 + R_2. \qquad (2.)$$

CASE II.—Let the load act *beyond* the points of support, as in fig. 89, which represents a cantilever or projecting beam, held up by a wall or other prop at S_1, held down by a notch in a mass of masonry or otherwise at S_2, and loaded so that P is the resultant of the load, including the weight of the beam. Then the proportional equation (1.) remains exactly as before; but the load is equal to the difference of the supporting pressures; that is to say,

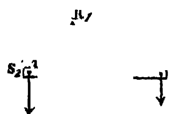


Fig. 89.

$$P = R_1 - R_2.$$

In these examples the beam is represented as horizontal; but the same principles would hold if it were inclined.

CASE III.—Let the directions of the supporting forces R_1, R_2, be now inclined to that of the resultant of the load, P, as in fig. 90. This case is that of the equilibrium of three forces treated of in Article 93, p. 136, and consequently the following principles apply to it:—

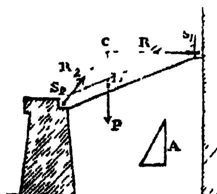


Fig. 90.

The lines of action of the supporting forces and of the resultant of the load must be in one plane.

They must intersect in one point (C, (fig. 90.

Those three forces must be proportional to the three sides of a triangle A, respectively parallel to their directions.

PROBLEM.—Given, the resultant of the load in magnitude and position, P, the line of action of one of the supporting forces, R_1, and the centre of resistance of the other, S_2; required, the line of action of the second supporting force, and the magnitudes of both.

Produce the line of action of R_1, till it cuts the line of action of P at the point C; join C S_2, this will be the line of action of R_2; construct a triangle Δ with its sides respectively parallel to those three lines of action; the ratios of the sides of that triangle will give the ratios of the forces.

To express this algebraically, let i_1, i_2, be the angles made by the lines of action of the supporting forces with that of the resultant of the load; then

$$P \cdot R_1 : R_2 : : \sin (i_1 + i_2) : \sin i_2 : \sin i_1. \text{ .... } (4.)$$

The same piece in a frame may act at once as a beam and tie, or as a beam and strut; or it may act alternately as a strut and as a tie, as the action of the load varies.

The load tends to break a tie by tearing it asunder, a strut by crushing it, and a beam by breaking it across. The power of materials to resist those tendencies will be considered in a later section.

113. **Distributed Loads.** (*A. M.*, 147.)—Before applying the principles of the present section to frames in which the load, whether external or arising from the weight of the bars, is distributed over their length, it is necessary to reduce that distributed load to an equivalent load, or series of loads, applied at the centres of resistance. The steps in this process are as follows:—

I. Find the resultant load on each single bar.

II. Resolve that load, as in Article 112, equation 1, p. 174, into two parallel components acting through the centres of resistance at the two ends of the bar.

III. At each centre of resistance where two bars meet, combine the component loads due to the loads on the two bars into one resultant, which is to be considered as the total load acting through that centre of resistance.

IV. When a centre of resistance is also a point of support, the component load acting through it, as found by step II. of the process, is to be left out of consideration until the supporting force required by the system of loads at the other joints has been determined; with this supporting force is to be compounded a force equal and opposite to the component load acting directly through the point of support, and the resultant will be the total supporting force.

In the following Articles of this section, all the frames will be

supposed to be loaded only at those centres of resistance which are *not* points of support; and therefore, in those cases in which components of the load act directly through the points of support also, forces equal and opposite to such components must be combined with the supporting forces as determined in the following Articles, in order to complete the solution.

114. **Frames of Two Bars.** (*A. M.*, 145–6.)—Figures 91, 92, and 93, represent cases in which a frame, of two bars jointed to each at the point L, is loaded at that point with a given force, P, and is

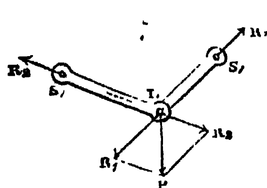


Fig 91.

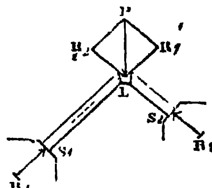


Fig 92.

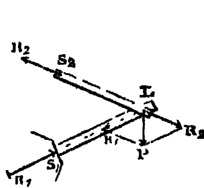


Fig 93.

supported by the connection of the bars at their farther extremities, S_1, S_2, with fixed bodies. It is required to find the stress on each bar, and the supporting forces at S_1 and S_2.

Resolve the load P (as in Article 94, p. 137) into two components, R_1, R_2, acting along the respective lines of resistance of the two bars. Those components are the loads borne by the two bars respectively; to which loads the supporting forces at S_1, S_2, are equal and directly opposed.

The symbolical expression of this solution is as follows:—Let i_1, i_2, be the respective angles made by the lines of resistance of the bars with the line of action of the load; then

$$P : R_1 : R_2 : : \sin (i_1 + i_2) : \sin i_2 : \sin i_1.$$

The inward or outward direction of the forces acting along each bar indicates that the stress is a thrust or a pull, and the bar a strut or a tie, as the case may be. Fig. 91 represents the case of two ties; fig. 92, that of two struts (such as a pair of rafters abutting against two walls); fig. 93 that of a strut, L S_1, and a tie, L S_2 (such as the jib and the tie-rod of a crane).

A frame of two bars is *stable* as regards deviations in the plane of its lines of resistance.

With respect to *lateral* deviations of angular position, in a direction perpendicular to that plane, a frame of two ties is stable; so also is a frame consisting of a strut and a tie, when the direction of the load inclines *from* the line S_1 S_2, joining the points of support

A frame consisting of a strut and a tie, when the direction of the load inclines *towards* the line $S_1 S_2$, and a frame of two struts in all cases, are unstable laterally, unless provided with lateral stays.

These principles are true of *any pair of adjacent bars whose farther centres of resistance are fixed;* whether forming a frame by themselves, or a part of a more complex frame.

115. **Triangular Frames.** (*A. M.*, 148, 149.)—Let fig. 94 represent a frame, consisting of the three bars A, B, C, connected at the three joints 1, 2, 3,—viz., C and A at 1, A and B at 2, B and C at 3. Let a load P_1 be applied at the joint 1 in any given direction; let supporting forces, P_2, P_3, be applied at the joints 2, 3; the lines of action of those two forces must be in the same plane with that of P_1, and must either be parallel to it or intersect it in one point. The latter case is taken first, because its solution comprehends that of the former.

Fig. 94.

The three external forces balance each other, and are therefore proportional to the three sides of a triangle respectively parallel to their directions. In fig. 95, let A B C be such a triangle, in which

$\overline{CA}$ represents P_1,
$\overline{AB}$ .. P_2,
$\overline{BC}$

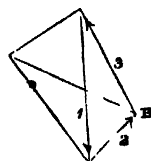


Fig. 95.

Draw C O parallel to the bar C, and A O parallel to the bar A, meeting in the point O, and join B O, which will be parallel to B.

The lengths of the three lines radiating from O will represent the stresses on the bars to which they are respectively parallel.

When the three external forces are parallel to each other, the triangle of forces A B C of fig. 95 becomes a straight line C A, as in fig. 96, divided into two segments by the point B. Let straight lines radiate from O to A, B, C, respectively parallel to the bars of the frame; then if the load C A be applied at 1 (fig. 94), A B applied at 2, and B C applied at 3, are the supporting forces required to balance it; and the radiating lines O A, O B, O C, represent the stresses on the bars A, B, C, respectively, as before.

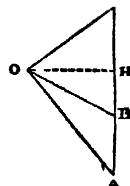


Fig. 96.

From O let fall O H perpendicular to C A, the common direction of the external forces. Then that line will represent a component of the stress, which is of equal amount in each bar. When C A, as is usually the case, is vertical, O H is horizontal; and the force represented by it is called

the "*horizontal thrust*" of the frame. *Horizontal Stress* or *Resistance* would be a more precise term; because the force in question is a pull in some parts of the frame, and a thrust in others.

In fig. 94, A and C are *struts*, and B a *tie*. If the frame were exactly inverted, all the forces would bear the same proportions to each other; but A and C would be *ties*, and B a *strut*.

The trigonometrical expression of the relations amongst the forces acting in a triangular frame, under parallel forces, is as follows:—

Let a, b, c, denote the respective angles of inclination of the bars A, B, C, to the line O H (that is, in general, to a horizontal line); then

$$\text{Horizontal Stress O H} = \frac{\text{load C A}}{\tan c : \cdot : \tan a} \;\ldots\ldots\; (1.)$$

$$\text{Supporting Forces} \begin{cases} \text{A B} = \text{O H} \cdot (\tan a \mp \tan b); \\ \text{B C} = \text{O H} \cdot (\tan b \pm \tan c). \end{cases} \;\ldots\; (2.)$$

The sign $\begin{Bmatrix} + \\ - \end{Bmatrix}$ is to be used when the two inclinations are in $\begin{Bmatrix} \text{opposite directions} \\ \text{the same direction.} \end{Bmatrix}$

$$\text{Stresses} \begin{cases} \text{O A} = \text{O H} \cdot \sec a \\ \text{O B} = \text{O H} \cdot \sec b \\ \text{O C} = \text{O H} \cdot \sec c \end{cases} \;\ldots\ldots\ldots\; (3.)$$

116. **Polygonal Frame.** (*A M.*, 150, 153.)— In fig. 97, let A, B, C, D, E, be the lines of resistance of the bars of a frame, connected together at the joints, whose centres of resistance are, 1 between A and B, 2 between B and C, 3 between C and D, 4 between D and E, and 5 between E and A. In the figure, the frame consists of five bars; but the principle is applicable to any number.

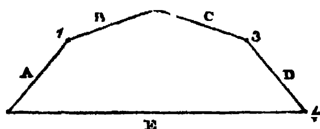


Fig 97.

From a point O, in fig. 98, (which may be called the *Diagram of Forces*), draw radiating lines O A, O B, O C, O D, O E, parallel respectively to the lines of resistance of the bars; and on those radiating lines take any lengths whatsoever, to represent the stresses on the several bars, which may have any magnitudes within the limits of strength of the material. Join the points thus found by straight lines, so as to form a closed polygon A B C D E A; then the sides of the polygon will represent a system of forces, which, being applied to the joints of the frame, will balance each other; each such force being applied to the joint between the bars whose lines of resistance are parallel to the pair

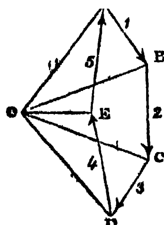


Fig. 98.

of radiating lines that enclose the side of the polygon of forces representing the force in question.

When the external forces are parallel to each other, the polygon of forces of fig. 98 becomes a straight line A D, as in fig. 99, divided into segments by the radiating lines; and each segment represents the external force which acts at the joint of the bars whose lines of resistance are parallel to the radiating lines that bound the segment. Moreover, the segment of the line A D which is intercepted between the radiating lines parallel to the lines of resistance of *any two bars*, *whether contiguous or not*, represents the resultant of the external forces which act at points *between the bars*.

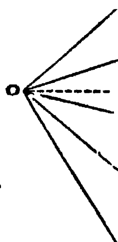


Fig. 99.

Thus, A D represents the total load, consisting of the three portions A B, B C, C D, applied at 1, 2, 3, respectively. D A represents the total supporting force, equal and opposite to the load, consisting of the two portions D E, E A, applied at 4 and 5 respectively. A C represents the resultant of the load applied between the bars A and C; and similarly for any other pair of bars.

From O draw O H perpendicular to A D; then that line represents a component of the stress, whose amount is the same in each bar of the frame. When the load, as is usually the case, is vertical, that component is called the "*horizontal thrust*" of the frame, and, as in Article 114, might more correctly be called *horizontal stress* or *resistance*, seeing that it is a pull in some of the bars and a thrust in others.

The trigonometrical expression of those principles is as follows:—

Let the force O H be denoted simply by H.

Let i, i', denote the inclinations to O H of the lines of resistance of *any two bars*, contiguous or not.

Let R, R', be the respective stresses which act along those bars.

Let P be the resultant of the external forces acting through the joint or joints between those two bars.

Then

$$P = H(\tan i \pm \tan i'); \ldots\ldots\ldots\ldots(1.)$$

$$R = H \cdot \sec i; \; R' = H \cdot \sec i'. \ldots\ldots\ldots\ldots(2.)$$

The $\left\{ \begin{matrix} \text{sum} \\ \text{difference} \end{matrix} \right\}$ of the tangents of the inclinations is to be used, according as they are $\left\{ \begin{matrix} \text{opposite} \\ \text{similar} \end{matrix} \right\}$.

117. **Open Polygonal Frame.** (*A.M.*, 151, 154.)—When the frame, instead of being closed, as in fig. 97, is converted into an open frame, by the omission of one bar, such as E, the corresponding modification

is made in the diagram of inclined forces, fig. 98, by omitting the lines O E, D E, E A, and in the diagram of parallel forces, fig. 99, by omitting the line O E. Then, in both diagrams, D O and O A represent the *supporting forces* respectively, equal and directly opposed to the stresses along the extreme bars of the frame, D and A, which must be exerted by the supports (called in this case *abutments*), at the points 4 and 5, against the ends of those bars, in order to maintain the equilibrium.

In the case of parallel loads, the following formulæ give the horizontal stress and supporting pressures.

Let i_d and i_a denote the angles of inclination of the bars D and A respectively.

Let $R_d = OD$ and $R_a = OA$ be the stresses along them.

Let $\Sigma \cdot P = AD$ denote the total load on the frame; then,

$$H = \frac{}{\tan i_d + \tan i_a}, \quad \ldots\ldots \tag{1}$$

$$R_d = H \cdot \sec i_d; \; R_a = H \cdot \sec i_a \tag{2.}$$

118. **Polygonal Frame—Stability.** (*A M.*, 152.)—The stability or instability of a polygonal frame depends on the principles stated in Article 112, p. 173, viz., that if a bar be free to change its angular position, then if it is a tie it is stable, and if a strut, unstable; and that a strut may be rendered stable by fixing its ends.

For example, in the frame of fig. 97, E is a tie, and stable; A, B, C, and D, are struts, free to change their angular position, and therefore unstable.

But these struts may be rendered stable in the plane of the frame by means of stays; for example, let two stay bars connect the joints 1 with 4, and 3 with 5, then the points 1, 2, and 3, are all fixed, so that none of the struts can change their angular positions. The same effect might be produced by two stay-bars connecting the joint 2 with 5 and 4.

The frame, as a whole, is unstable, as being liable to overturn laterally, unless provided with lateral stays, connecting its joints with fixed points.

Now, suppose the frame to be exactly inverted, the loads at 1, 2, and 3, and the supporting forces at 4 and 5, being the same as before. Then E becomes a strut; but it is stable, because its ends are fixed in position; and A, B, C, and D becomes ties, and are stable without being stayed.

An open polygon consisting of ties, such as is formed by A, B, C, and D, when inverted, is called by mathematicians, a *funicular polygon*, because it may be made of ropes.

It is to be observed, that the stability of an *unstayed* polygon of

ties is of the kind which admits of *oscillation* to and fro about the position of equilibrium. That oscillation may be injurious in practice, and stays may be required to prevent it.

119. **Bracing of Frames.** (*A. M.*, 155.)—A *brace* is a stay-bar on which there is a permanent stress. If the distribution of the loads on the joints of a polygonal frame, though consistent with its equilibrium as a whole, be not consistent with the equilibrium of each bar, then, in the diagram of forces, when converging lines respectively parallel to the lines of resistance are drawn from the angles of the polygon of external forces, those converging lines,* instead of meeting in one point, will be found to have gaps between them. The lines necessary to fill up those gaps will indicate the forces to be supplied by means of the resistance of braces.*

The resistance of a brace introduces a pair of equal and opposite forces, acting along the line of resistance of the brace, upon the pair of joints which it connects. It therefore does not alter the *resultant* of the forces applied to that pair of joints in amount nor in position, but only the *distribution* of the components of that resultant on the pair of joints.

To exemplify the use of braces, and the mode of determining the stresses on them, let fig. 100 represent a frame such as frequently

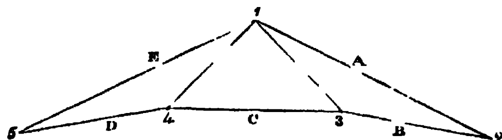


Fig. 100.

occurs in iron roofs, consisting of two struts or rafters, A and E, and three tie-bars, B, C, and D, forming a polygon of five sides, jointed at 1, 2, 3, 4, 5, loaded vertically at 1, and supported by the vertical resistance of a pair of walls at 2 and 5. The joints 3 and 4 having no loads applied to them, are connected with 1 by the braces 1 4 and 1 3.

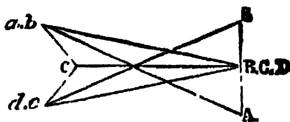


Fig. 101.

To make the diagram of forces (fig. 101), draw the vertical line E A, as in Article 116, to represent the direction of the load and of the supporting forces.

* This method of treating *braced* frames contains an improvement suggested by Mr. Clerk Maxwell in 1867.

The two segments of that line, A B and D E, are to be taken to represent the supporting forces at 2 and 5; and the whole line E A will represent the load at 1. From the ends, and from the point of division of the *scale of external forces*, E A, draw straight lines parallel respectively to the lines of resistance of the frame, each line being drawn from the point in E A that is marked with the corresponding letter. Then A *a* and B *b*, meeting at *a*, *b*, will represent the stresses along A and B respectively; and E *e* and D *d*, meeting in *d*, *e*, will represent the stresses along D and E respectively; but those four lines, instead of meeting each other and C *c* parallel to C in one point, leave *gaps*, which are to be filled up by drawing straight lines parallel *to the braces:* that is to say, from *a*, *b*, to *c*, parallel to 1 3; and from *d*, *e*, to *c* parallel to 4 1. Then those straight lines will represent the stresses along the braces to which they are respectively parallel; and C *c* will represent the tension along C. To each joint in the frame, fig. 100, there corresponds, in fig. 101, a triangle, or other closed polygon, having its sides respectively parallel, and therefore proportional, to the forces that act at that joint. For example,

Joints,	1,	2,	3,	4,	5,
Polygons,	E A *a c e* E;	A B *b* A;	B *c b* B;	D *d c* D;	D E *e* D.

The order of the letters indicates the directions in which the forces act relatively to the joints.

Another method of treating simple cases of bracing is illustrated by fig. 102. A and B are two struts, forming the two halves of

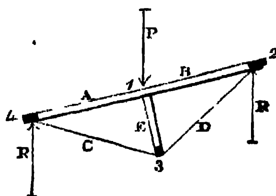


Fig. 102.

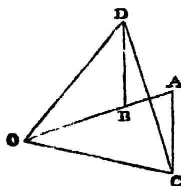


Fig. 103.

one straight bar; C and D are two equal tie-rods; E, a strut-brace. A vertical load P is applied at the joint 1, between A and B; two vertical supporting pressures, each denoted by R = P ÷ 2, act at the joints 4 and 2. The joint 3 has no external load.

Fig. 103 is the diagram of forces, constructed as follows:—Through a point O draw O B A parallel to A and B, O C parallel

to C, and O D parallel to D. Make O D = O C; join C D; this line will be parallel to the brace E, and perpendicular to O A.

Through D and C draw vertical lines D B, C A; these, being equal to each other, are to be taken to represent the two supporting pressures R; and their sum D B + A C will represent the load P. The equal tensions on C and D will be represented by O C and O D, and the thrusts along A, B, and E, by O A, O B, and C D.

The polygon of external forces in this case is the crossed quadrilateral A C D B, in which C A and B D represent (as already stated) the supporting pressures, and D C and A B the components of the load P respectively parallel and perpendicular to the brace E. When A and B are horizontal, and E vertical, A B in fig. 103 vanishes, and B D and C A coincide with the two halves of C D.

120. **Rigidity of a Truss.** (*A. M.*, 156, 157.)—The word *truss* is applied in carpentry to a triangular frame, and to a polygonal frame to which rigidity is given by staying and bracing, so that its figure shall be incapable of alteration by turning of the bars about their joints. If each joint were *absolutely* of the kind described in Article 111, that is, like a hinge, incapable of offering any resistance to alteration of the relative angular position of the bars connected by it, it would be necessary, in order to fulfil the condition of rigidity, that every polygonal frame should be divided by the lines of resistance of stays and braces into triangles and other polygons, so arranged that every polygon of four or more sides should be surrounded by triangles on all but two sides and the included angle at farthest. For every unstayed polygon of four sides or more, with flexible joints, is flexible, unless all the angles except one be fixed by being connected with triangles.

Sometimes, however, a certain amount of stiffness in the joints of a frame, and sometimes the resistance of its bars to bending, is relied upon to give rigidity to the frame, when the load upon it is subject to small variations only in its mode of distribution. For example, in the truss of fig. 104, the tie-beam A A is made in one piece, or in two or more pieces so connected together as to act like one piece; and part of its weight is suspended from the joints C, C, by the rods C B, C B. These rods also serve to make the resistance of the tie-beam A A to being bent act so as to prevent the struts A C, C C, C A, from deviating from their proper angular positions, by turning on the joints A, C, C, A. If A B, B B, and B A, were three distinct pieces, with flexible joints at B B, it is evident that the frame might be disfigured by distortion of the quadrangle B C C B.

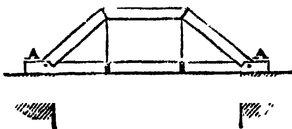


Fig. 104.

The object of stiffening a truss by braces is to enable it to sustain loads variously distributed; for were the load always distributed in one way, a frame might be designed of a figure exactly suited to that load, so that there should be no need of bracing.

The variations of load produce variations of stress on all the pieces of the frame, but especially on the braces; and each piece must be suited to withstand the greatest stress to which it is liable.

Some pieces, and especially braces, may have to act sometimes as struts and sometimes as ties, according to the mode of distribution of the load.

121. **Secondary and Compound Trussing.** (*A. M.*, 158 to 160.)—A *secondary truss* is a truss which is supported by another truss.

When a load is distributed over a great number of centres of resistance, it may be advantageous, instead of connecting all those centres by one polygonal frame, to sustain them by means of several small trusses, which are supported by larger trusses, and so on, the whole structure of secondary trusses resting finally on one large truss, which may be called the *primary truss.* In such a combination the same piece may often form part of different trusses; and then the stress upon it is to be determined according to the following principle:—

When the same bar forms at the same time part of two or more different frames, the stress on it is the resultant of the several stresses to which it is subject by reason of its position in the several frames.

In a *Compound Truss*, several frames, without being distinguishable into primary and secondary, are combined and connected in such a manner that certain pieces are common to two or more of them, and require to have their stresses determined by the principle above stated.

Examples of secondary and compound trusses will be given in treating of structures in timber and iron.

122. **Resistance of a Frame at a Section.** (*A. M.*, 161.)—The labour of calculating the stress on the bars of a frame may sometimes be abridged by the application of the following principle:—

If a frame be acted upon by any system of external forces, and if that frame be conceived to be completely divided into two parts by an ideal surface, the stresses along the bars which are intersected by that surface, balance the external forces which act on each of the two parts of the frame.

In most cases which occur in practice, the lines of resistance of the bars, and the lines of action of the external forces, are all in one vertical plane, and the external forces are vertical. In such cases the most convenient position for an assumed plane of section is vertical, and perpendicular to the plane of the frame. Take the vertical line of intersection of these two planes for an axis of co-

ordinates,—say for the axis of y, and any convenient point in it for the origin O; let the axis of x be horizontal, and in the plane of the frame, and the axis of z horizontal, and in the plane of section.

The external forces applied to the part of the frame at one side of the plane of section (either may be chosen) being combined, as in Article 99, p. 146, give three data—viz., the total force along $x = \Sigma \cdot X$; the total force along $y = \Sigma \cdot Y$; and the moment of the couple acting round $z = M$; and the bars which are cut by the plane of section must exert resistances capable of balancing those two forces and that couple. If not more than three bars are cut by the plane of section, there are not more than three unknown quantities, and three relations between them and given quantities, so that the problem is determinate; if more than three bars are cut by the plane of section, the problem is or may be indeterminate.

The formulæ to which this reasoning leads are as follows:—Let x be positive in a direction from the plane of section towards the part of the structure which is considered in determining $\Sigma \cdot X$, $\Sigma \cdot Y$, and M; let $+y$ be measured upwards; let angles measured from O x towards $+y$, that is, upwards, be positive; and let the lines of resistance of the three bars cut by the plane of section make the angles i_1, i_2, i_3, with x. Let n_1, n_2, n_3, be the perpendicular distances of those three lines of resistance from O, distances lying $\left.\begin{matrix}\text{upwards}\\ \text{downwards}\end{matrix}\right\}$ from O x being considered as $\left\{\begin{matrix}\text{positive}\\ \text{negative}\end{matrix}\right\}$.

Let R_1, R_2, R_3, be the resistances, or total stresses, along the three bars, pulls being positive, and thrusts negative. Then we have the following three equations:—

$$\left.\begin{aligned} \Sigma \cdot X &= R_1 \cos i_1 + R_2 \cos i_2 + R_3 \cos i_3; \\ \Sigma \cdot Y &= R_1 \sin i_1 + R_2 \sin i_2 + R_3 \sin i_3; \\ -M &= R_1 n_1 + R_2 n_2 + R_3 n_3; \end{aligned}\right\} \dots\dots(1.)$$

from which the three quantities sought, R_1, R_2, R_3, can be found.

Speaking with reference to the given plane of section, $\Sigma \cdot X$ may be called the *normal stress*, $\Sigma \cdot Y$, the *shearing stress*, and M, the *moment of flexure*, or *bending stress;* for it tends to bend the frame at the section under consideration. M is to be considered as $\left\{\begin{matrix}\text{positive}\\ \text{negative}\end{matrix}\right\}$ according as it tends to make the frame become concave $\left\{\begin{matrix}\text{upwards}\\ \text{downwards.}\end{matrix}\right\}$

Examples of the application of this method will be given in treating of lattice-beams of timber and iron.

123. **Balance of a Chain or Cord.**—A loaded chain may be looked upon as a polygonal frame whose pieces and joints are so numerous that its figure may without sensible error be treated as a continuous

curve. The following are the principles respecting the equilibrium of loaded chains and cords which are of most importance in practice.

I. *Balance of a Chain in general.*—Let D A C, in fig. 105, represent a flexible cord or chain supported at the points C and D, and loaded by forces, in any direction, constant or varying, distributed over its whole length with constant or varying intensity.

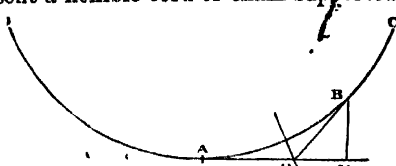


Fig. 105.

Let A and B be any two points in this chain; from those points draw tangents to the chain, A P and B P, meeting in P. The load acting on the chain between the points A and B is balanced by the pulls along the chain at those two points respectively; those pulls must respectively act along the tangents A P, B P; hence the resultant of the load between A and B acts through the point of intersection of the tangents at A and B; and that load, and the tensions on the chain at A and B, are respectively proportional to the sides of a triangle parallel to their directions.

II. *Chain under Vertical Load—Curve of Equilibrium.*—If the direction of the load be everywhere parallel and vertical, draw a vertical straight line, C D, fig. 106, to represent the total load, and from its ends draw C O and D O, parallel to two tangents at the points of support of the chain, and meeting in O; those lines will represent the tensions on the chain at its points of support.

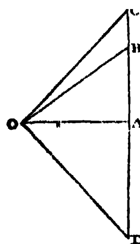


Fig. 106.

Let A, in fig. 105, be the lowest point of the chain. In fig. 106, draw the horizontal line O A; this will represent the horizontal component of the tension of the chain at every point, and if O B be parallel to a tangent to the chain at B (fig. 105) A B will represent the portion of the load supported between A and B, and O B the tension at B.

To express this algebraically, let

H = O A = *horizontal tension* along the chain at A;

R = O B = pull along the chain at B;

P = A B = load on the chain between A and B;

$i = \angle$ X P B (fig 105) $= \angle$ A O B (fig. 106) = inclination of chain at B;

then,

$$P = H \tan i; \; R = \sqrt{(P^2 + H^2)} = H \sec i \ldots\ldots\ldots (1.)$$

To deduce from these formulæ an equation by which the form of the curve assumed by the chain can be determined when the distribution of the load is known, let that curve be referred to rectangular horizontal and vertical co-ordinates, measured from the lowest point A, fig. 105, the co-ordinates of B being, A X = x, X B = y; then

$$\tan i = \frac{dy}{dx} = \frac{P}{H}. \quad \text{(2.)}$$

a differential equation which enables the form assumed by the cord (or "*curve of equilibrium*") to be determined when the distribution of the load is known.

124. **To Draw a Curve of Equilibrium approximately.**

Problem.—Let H and K, fig. 107, be the two points of suspension of a chain under a vertical load; let the distribution of the

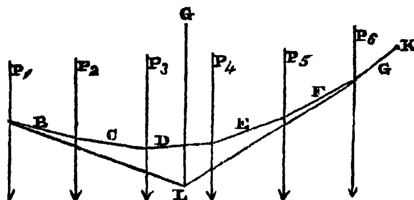


Fig. 107.

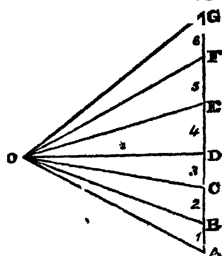


Fig. 108.

load be given, and the direction of a tangent H L to the chain at one of the points of suspension; it is required to draw approximately the figure of the chain.

Find the centre of gravity of the entire load, and let G L be a vertical line passing through it, cutting the tangent H L in L. Join K L; this will be the tangent to the chain at the other point of suspension.

Conceive the load to be divided into any convenient number of portions: the more numerous these are, the closer will be the approximation to the required curve. Find the centres of gravity of those portions, and let P_1, P_2, &c., be vertical lines passing through those centres of gravity.

In fig. 108, draw the vertical line, or *scale of loads*, A G, whose whole length represents the entire load; divide it into parts, A B or 1, B C or 2, &c., representing the several portions of the load. Through A draw A O parallel to L H, and through G draw G O parallel to K L, cutting each other in O. From O draw radiating lines O B, O C, &c., to the points of division of the scale of loads.

Then, in fig. 107, from the point of intersection of A, or H L,

with P_1, draw B parallel to O B, cutting P_2; from the point of intersection of B and P_2, draw C parallel to O C, cutting P_3, and so on, until the "funicular polygon" A B C, &c., is completed; that polygon is composed of tangents to the required curve of equilibrium, to which an approximation may be drawn by sketching a curve so as to touch the sides of the polygon.

125. **Chain under an Uniform Vertical Load—Suspension Bridge.** (*A. M.*, 169, 170.)—By an uniform vertical load is meant a load uniformly distributed along a horizontal straight line; so that if

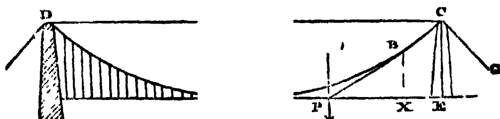


Fig. 109.

A, fig. 109, be the lowest point of the rope or cord, the load suspended between A and B shall be proportional to $\overline{AX} = x$, the horizontal distance between those points, and capable of being expressed by the equation

$$P = p\,x; \qquad (1.)$$

where p is a constant quantity, denoting the *intensity of the load in units of weight per unit of horizontal length:* in pounds per lineal foot, for example.

In this case, because the load between A and B is uniformly distributed, its resultant bisects A X; also, the tangent B P bisects A X; and the curve assumed by the chain is a PARABOLA whose vertex is at A.

The proportions of the load, and the horizontal and oblique tensions are as follows:—

$$\left.\begin{array}{l} P : H : R :: \overline{BX} : \overline{XP} : \overline{PB} :: y : \dfrac{x}{2} : \sqrt{\left(y^2 + \dfrac{x^2}{4}\right)} \\ \qquad :: p\,x : \dfrac{p\,x^2}{2\,y} : p\,x \cdot \sqrt{\left(1 + \dfrac{x^2}{4\,y^2}\right)}. \end{array}\right\} \quad (2.)$$

The focal distance of the parabola is

$$m = \frac{x^2}{4\,y_1} = \frac{H}{2\,p}. \qquad (3.)$$

These equations are applicable, with sufficient accuracy for practical purposes, to most examples of *Suspension Bridges with vertical rods;* for although in a bridge of that class the load is not continuous, the platform being hung by rods from a certain number of

points in each cable or chain; nor uniformly disturbed, the load arising from the weight of the cables or chains and of the suspending rods being more intense near the piers; yet, in most cases which occur in practice, the condition of each cable or chain approaches sufficiently near to that of a cord continuously and uniformly loaded to enable the preceding equations to be applied without material error.

The following solutions of some useful problems are deduced from these equations:—

PROBLEM FIRST.—*Given the elevations,* y_1, y_2, *of the two points of support of the chain above its lowest point, and also the horizontal distance, or span* a, *between those points of support: it is required to find the horizontal distances,* x_1, x_2, *of the lowest point from the two points of support: also the focal distance* m.

$$x_1 = a \cdot \frac{\sqrt{y_1}}{\sqrt{y_1} + \sqrt{y_2}}; \quad x_2 = a \cdot \frac{\sqrt{y_2}}{\sqrt{y_1} + \sqrt{y_2}} \ldots\ldots\ldots (4.)$$

$$m = \frac{\quad}{4\,y_1 + 4\,y_2 + 8\,\sqrt{y_1\,y_2}} \ldots\ldots\ldots\ldots \quad (5.)$$

When the points of support are at the same level,

$$y_1 = y_2; \; x_1 = \frac{a}{2}; \; m = \frac{a^2}{16\,y_1} \ldots\ldots\ldots\ldots\ldots\ldots (6.)$$

PROBLEM SECOND.—*Given the same data, to find the inclinations* i_1, i_2, *of the chain at the points of support.*

$$\tan i_1 = \frac{2\,y_1}{x_1} = \frac{2\,y_1 + 2\,\sqrt{y_1 y_2}}{a}; \; \tan i_2 = \frac{2\,y_2}{x_2} = \frac{2\,y_2 + 2\,\sqrt{y_1 y_2}}{a} \; (7.)$$

when $$y_1 = y_2, \; \tan i_1 = \tan i_2 = \frac{4\,y_1}{a} \ldots\ldots\ldots\ldots\ldots (8.)$$

PROBLEM THIRD.—*Given the same data, and the load* p *per unit of length: required the horizontal tension* H, *and the tensions* R_1, R_2, *at the points of support.*

$$H = 2\,p\,m = \frac{p\,a^2}{2\,y_1 + 2\,y_2 + 4\,\sqrt{y_1\,y_2}}; \ldots\ldots\ldots (9.)$$

$$R_1 = H\sqrt{\left(1 + \frac{4\,y_1^2}{x_1^2}\right)}; \; R_2 = H\sqrt{\left(1 + \frac{4\,y_2^2}{x_2^2}\right)} \ldots. (10.)$$

When $y_1 = y_2$, those equations become

$$H = \frac{p\,a^2}{8\,y_1};\ R_1 = R_2 = H\sqrt{\left(1 + \frac{16\,y_1^2}{a^2}\right)}. \ \ldots\ldots (11.)$$

Problem Fourth.—*Given the same data as in Problem First, to find the length of the chain.*

The following are two well-known formulæ for the length of a parabolic arc, commencing at the vertex, one being in terms of the co-ordinates x and y of the farther extremity of the arc, and the other in terms of the focal distance m, and the inclination i of the farther extremity of the arc to a tangent at the vertex.

$$s = \sqrt{\left(y^2 + \frac{x^2}{4}\right)} + \frac{x^2}{4\,y}\cdot \text{hyp. log.}\ \frac{y + \sqrt{\left(y^2 + \frac{x^2}{4}\right)}}{\frac{x}{2}}$$

$$= m\left\{\tan i \cdot \sec i + \text{hyp. log. } (\tan i + \sec i)\right\} \ \ldots\ldots (12.)$$

The *length of the chain* is $s_1 + s_2$, where s_1 is found by putting x_1 and y_1 in the first of the above formulæ, or i_1 in the second, and s_2 by putting x_2 and y_2 in the first formula, or i_2, in the second.

The following *approximate formula* for the length of a parabolic arc is in many cases sufficiently near the truth for practical purposes:

$$s = x + \frac{2}{3}\frac{y^2}{x}\ \textit{nearly}, \ \ldots\ldots\ldots\ldots\ldots\ldots (13.)$$

which gives the total length of the cord,

$$s_1 + s_2 = a + \frac{2}{3}\left(\frac{y_1^2}{x_1} + \frac{y_2^2}{x_2}\right)\textit{nearly}, \ \ldots\ldots\ldots\ldots (14.)$$

and when $y_1 = y_2$, this becomes

$$2\,s_1 = a + \frac{8}{3}\cdot\frac{y_1^2}{a}\ \textit{nearly}, \ \ldots\ldots\ldots\ldots\ldots\ldots (15.)$$

Problem Fifth.—*Given the same data, to find, approximately, the small elongation of the chain* d $(s_1 + s_2)$ *required to produce a given small depression* d y *of the lowest point* A, *and conversely.*

$$\frac{d\,(s_1 + s_2)}{d\,y} = \frac{4}{3}\left(\frac{y_1}{x_1} + \frac{y_2}{x_2}\right) \ldots\ldots\ldots\ldots\ldots\ldots (16.)$$

When $y_1 = y_2$, this equation becomes

$$\frac{2\,d\,s_1}{d\,y} = \frac{16\,y_1}{3\,a} \ldots\ldots\ldots\ldots\ldots\ldots\ldots \text{(17.)}$$

These formulæ serve to compute the depression which the middle point of a suspension bridge undergoes in consequence of a given elongation of the cable or chain, whether caused by heat or by tension.

PROBLEM SIXTH.—*To find the pressure on the top of each pier.*

When the piers of a suspension bridge are slender and vertical (as is usually the case), the resultant pressure of the chain or cable on the top of the pier ought to be vertical also. Thus, let C E, in fig. 109, represent the vertical axis of a pier, and C G the portion of the chain or cable behind the pier, which either supports another division of the platform, or is made fast to a mass of rock, or of masonry, or otherwise. If the chain or cable passes over a curved plate on the top of the pier called a *saddle*, on which it is free to slide, the tensions of the portions of the chain or cable on either side of the saddle will be equal; and in order that those tensions may compose a vertical pressure on the pier, their inclinations must be equal and opposite. Let i be the common value of those inclinations; R the common value of the two tensions; then the vertical pressure on the pier is

$$\mathrm{V} = 2\,\mathrm{R} \sin i = 2\,\mathrm{H} \tan i = 2\,p\,x; \ \ldots\ldots\ldots \text{(18.)}$$

that is, twice the weight of the portion of the bridge between the pier and the lowest point, A, of the curve C B A D.

But if the two divisions of the chain or cable D A C, C G, which meet at C, be *made fast* to a sort of truck, which is supported by rollers on a *horizontal* cast iron platform on the top of the pier, then the pressure on the pier will be vertical, whether the inclinations of the two divisions of the chain or cable be equal or unequal; and it is only necessary that the *horizontal components* of their tension should be equal; that is to say, let i, i', be the inclinations of the two divisions of the chain or cable in opposite directions at C, and R, R′, their tensions, then

$$\mathrm{R} = \mathrm{H} \sec i; \ \mathrm{R}' = \mathrm{H} \sec i';$$

$$\mathrm{V} = \mathrm{R} \sin i + \mathrm{R}' \sin i' = \mathrm{H}\,(\tan i + \tan i') \ldots\ldots \text{(19.)}$$

126. **Suspension Bridge with Sloping Rods.** (*A. M.*, 172.)—Let the uniformly-loaded platform of a suspension bridge be hung from the chains by parallel sloping rods, making an uniform angle j with the vertical. The condition of a chain thus loaded is the same with that of a chain loaded vertically, except in the direction of the

load; and the form assumed by the chain is a parabola, having its axis parallel to the direction of the suspension-rods.

In fig. 110, let C A represent a chain, or portion of a chain, supported or fixed at C, and horizontal at A, its lowest point. Let A H be a horizontal tangent at A, representing the platform of the bridge; and let the suspension rods be all parallel to C E, which makes the angle $\angle$ E C H $= j$ with the vertical. Let B X represent any rod, and suppose a vertical load v to be supported at the point X.

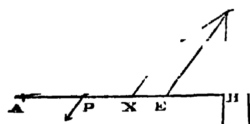


Fig. 110.

Then, by the principles of the equilibrium of a *frame of two bars*, this load will produce a *pull*, p, on the rod X B, and a *thrust*, q, on the platform between X and H; and the three forces v, p, q, will be proportional to the sides of a triangle parallel to their directions, such as the triangle C E H; that is to say,

$$v : p : q :: \overline{\mathrm{CH}} : \overline{\mathrm{CE}} : \overline{\mathrm{EH}} :: 1 : \sec j : \tan j \text{ ..... (1.)}$$

Next, instead of considering the load on one rod B X, consider the entire vertical load V between A and X.

Let P represent the amount of the pull acting on the rods between A and X, and Q the total thrust on the platform at the point X; then,

$$\mathrm{V} : \mathrm{P} : \mathrm{Q} :: \overline{\mathrm{CH}} : \overline{\mathrm{CE}} : \overline{\mathrm{EH}} :: 1 : \sec j : \tan j \text{... (2.)}$$

The *oblique load* $\mathrm{P} = \mathrm{V} \sec j$ is what hangs from the chain between A and B. Being uniformly distributed, its resultant bisects A X in P, which is also the point of intersection of the tangents A P, B P; and the ratio of the oblique load P, the horizontal tension H along the chain at A, and the tension R along the chain at B, is that of the sides of the triangle B X P; that is to say,

$$\mathrm{P} : \mathrm{H} : \mathrm{R} :: \overline{\mathrm{BX}} : \overline{\mathrm{XP}} = \frac{\overline{\mathrm{AX}}}{2} : \overline{\mathrm{BP}} \text{......... (3.)}$$

The curve C B A is a parabola having its axis parallel to the inclined suspension rods; and its equation referred to oblique co-ordinates, with the origin at A, is as follows. Let A X $= x$, X B $= y$; then,

$$y = \frac{x^2 \cdot \cos^2 j}{4m} \text{ .... (4.)}$$

where m, as in Article 125, denotes the focal distance of the parabola, given by the equation

$$m = \frac{x^2 \cdot \cos^2 j}{4\, y} \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (5.)$$

x and y being the co-ordinates of any *known* point in the curve. The length of the tangent $\overline{\mathrm{B\,P}} = t$ is given by the following equation:—

$$t = \sqrt{\left(\frac{x^2}{4} + y^2 + x\,y \cdot \sin j\right)} \ldots\ldots\ldots\ldots\ldots (6.)$$

Hence are deduced the following formulæ for the relations amongst the forces which act in a suspension bridge with inclined rods:—Let v now be taken to denote the *intensity* of the vertical load per unit of length of horizontal platform—per foot, for example; p the intensity of the oblique load; q the rate at which the thrust along the platform increases from A towards H. Then

$$\left.\begin{aligned} \mathrm{V} &= v\,x; \\ \mathrm{P} &= p\,x = v\,x \cdot \sec j; \\ \mathrm{Q} &= q\,x = v\,x \cdot \tan j; \end{aligned}\right\} \ldots\ldots\ldots\ldots\ldots\ldots (7.)$$

$$\mathrm{H} = \frac{x\,\mathrm{P}}{2y} = \frac{p\,x^2}{2y} = \frac{2\,p\,m}{\cos^2 j} = 2\,v\,m \cdot \sec^3 j \ldots\ldots\ldots (8.)$$

$$\mathrm{R} = \frac{t\,\mathrm{P}}{y} = \frac{2\,t\,\mathrm{H}}{x} = \frac{p\,t\,x}{y} = \frac{v\,t\,x \sec j}{y} \ldots\ldots\ldots\ldots (9.)$$

The horizontal pull H at the point A may be sustained in three different ways, viz.:—

I. The chain may be *anchored* or made fast at A to a mass of rock or masonry.

II. It may be attached at A to another equal and similar chain, similarly loaded by means of oblique rods, sloping at an equal angle in the direction opposite to that of the rods B X, &c., so that A may be in the middle of the span of the bridge.

III. The chain may be made fast at A to the horizontal platform A H, so that the pull at A shall be balanced by an equal and opposite thrust along the platform, which must be strong enough and stiff enough to sustain that thrust. In this case, the total thrust at any point, X, of the platform is no longer simply $\mathrm{Q} = q\,x$, but

$$\begin{aligned} \mathrm{H} + \mathrm{Q} &= \left(\frac{\mathrm{P}}{2\,y} + q\right) x \\ &= v\,(2\,m \cdot \sec^3 j + x \cdot \tan j) \ldots\ldots\ldots\ldots\ldots (10.) \end{aligned}$$

The *length of the parabolic arc*, A B, is given exactly by the

following formulæ:—Let i denote the inclination of the parabola at the point B to a line perpendicular to its axis. Then

$$i = \text{arc} \cdot \cos\left(\frac{x}{2t} \cdot \cos j\right) \text{ ....(11.)}$$

which, when B coincides with A, becomes simply $i = j$. Then from the known formulæ for the lengths of parabolic arcs, we have

$$\text{parabolic arc A B} = m \left\{ \tan i \sec i - \tan j \sec j \right.$$

$$\left. + \text{hyp. log.} \frac{\tan i + \sec i}{\tan j + \sec j} \right\} \text{ ....(12.)}$$

In most cases which occur in practice, however, it is sufficient to use the following approximate formula:—

$$\text{arc A B} = x + y \cdot \sin j + \frac{2}{3} \cdot \frac{y^2 \cdot \cos^2 j}{x + y \cdot \sin j}, \text{ nearly. .....(13.)}$$

The formulæ of this Article are applicable to Mr. Dredge's suspension bridges, in which the suspending rods are inclined, and although not exactly parallel, are nearly so.

127. **Deflection of a Flexible Tie.** (*A. M.*, 171.)—Let a vertical load, P, be applied at A, fig. 111, and sustained by means of a horizontal strut, A B, abutting against a pier at B, and a sloping rope or chain, or other flexible tie, A D C, fixed to the top of the pier at C. The weight of the strut, A B, is supposed to be divided into two components, one of which is supported at B, while the other is *included* in the load P. The weight, W, of the flexible tie, A D C, is supposed to be known, and to be considered separately; and with these data there is proposed the following

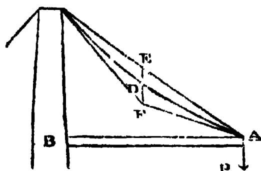


Fig. 111.

PROBLEM.—W *being small compared with* P, *to find approximately the vertical depression* E D *of the flexible tie below the straight line* A C, *the pulls along it at* A, D, *and* C, *and the horizontal thrust along* A B.

Because W is small compared with P, the curvature of the tie will be small, and the distribution of its weight along a horizontal line may be taken as *approximately* uniform; therefore its figure will be *nearly* a parabola; the tangent at D will be sensibly parallel to A C, and the tangents at A and C will meet in a point which will be near the vertical line E D F, which line bisects A C, and is

bisected in D. Hence the following solution is in general sufficiently near the truth for practical purposes. Let R_a, R_d, R_c, be the tensions of the tie at A, D, C, respectively, and H the horizontal thrust; then

$$H = \left(P + \frac{W}{2}\right)\frac{\overline{AB}}{\overline{BC}};$$

$$R_a = \sqrt{\left(H^2 + P^2\right)};$$

$$R_d = \sqrt{\left\{H^2 + \left(P + \frac{W}{2}\right)^2\right\}} \qquad (1.)$$

$$R_c = \sqrt{\left\{H^2 + (P + W)^2\right\}};$$

$$\overline{DE} = \tfrac{1}{4}\,\overline{BC}\,\frac{W}{P + \frac{W}{2}}.$$

The *difference of length* between the curve A D C and the straight line A E C is found very nearly by the following formula:—

$$\overline{ADC} - \overline{AEC} = \frac{2}{3}\cdot\frac{\overline{AB}^2\cdot\overline{DE}^2}{\overline{AC}^3} = \frac{1}{24}\,\frac{\overline{AB}^2\cdot\overline{BC}^2}{\overline{AC}^3}\left(\frac{W}{P + \frac{W}{2}}\right)^2 \ldots (2.)$$

If E F be made = 2 D E, F C and F A will be approximately tangents to the chain at C and A.

128. The **Catenary** (*A. M.*, 175), in the most general sense of the word, is the curve formed by a chain when loaded in any manner; but when used without qualification, its application is usually restricted to the case of a chain of uniform section and material, loaded with its own weight only. As thus defined, the catenary has the following properties:—

I. All catenaries are similar.

II. The *figure* of the catenary is expressed algebraically by the following equation. (See fig. 112.) Let A be the *vertex*, or lowest point of the catenary, where it is horizontal. A O is a vertical line, called the *parameter*, or *modulus* of the catenary, on which all its dimensions depend; let the length of that line be denoted by m. Take O for the origin of co-ordinates. Let B be any other point in the catenary, whose abscissa

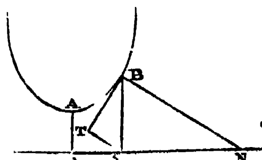


Fig. 112.

or horizontal distance from O, is O X $= x$, and vertical height above the same point X B $= y$. Then

$$\text{the ordinate X B} = y = \frac{m}{2}\left(e^{\frac{x}{m}} + e^{-\frac{x}{m}}\right);^{*}$$

$$\text{the arc A B} = s = \frac{m}{2}\left(e^{\frac{x}{m}} - e^{-\frac{x}{m}}\right) = \sqrt{y^2 - m^2};$$

the abscissa in terms of the ordinate,

$$x = m \;\; \text{hyp. log.} \left(\frac{y}{m} + \sqrt{\frac{y^2}{m^2} - 1}\right); \qquad (1.)$$

$$\text{the area, A O X B} = \int y\, dx = \frac{m^2}{2}\left(e^{\frac{x}{m}} - e^{-\frac{x}{m}}\right) = ;$$

the rate of slope at the point B,

$$\frac{d\,y}{d\,x} = \tan i = \frac{1}{2}\left(e^{\frac{x}{m}} - e^{-\frac{x}{m}}\right) = \frac{s}{m};$$

(i being the angle of inclination of the curve at B to the horizon). The radius of curvature at the same point is,

$$\rho = \frac{y^2}{m} = \frac{m}{4}\left(e^{\frac{2x}{m}} + e^{-\frac{2x}{m}} + 2\right);$$

at the point A, $\rho = m$.

On B X as a hypothenuse construct the right-angled triangle X T B, in which X T $=$ O A $= m$. Then T B will be a tangent to the catenary at B, and will be equal in length to the arc A B $= s$.

Through B draw B N perpendicular to B T, cutting O X produced in N; B N is equal to the radius of curvature at the point B.

III. The *mechanical* properties of the catenary are as follows:—

Let p be the weight of an unit of length of the chain (as one foot);

* The functions $e^{\frac{x}{m}}$ and $e^{-\frac{x}{m}}$, or the *Naperian Anti-logarithm* of $\frac{x}{m}$ and its reciprocal, are most easily calculated by means of a table of Naperian or hyperbolic anti-logarithms, or, in its absence, by a table of hyperbolic logarithms. But should a table of common logarithms or anti-logarithms alone be at hand, the following formula is to be used:—

$$10^{\cdot 4343\,\frac{x}{m}}, \text{ nearly.}$$

H, the horizontal tension at A; P, the vertical load between the points A and B; R, the tension at B. Then

$$\mathrm{H} = p\,m;\ \mathrm{P} = p\,s;\ \mathrm{R} = \sqrt{\mathrm{H}^2 + \mathrm{P}^2} = p\,y;\ \ldots\ldots\ (2.).$$

so that the parameter represents a length of chain whose weight is equal to the horizontal tension; and the ordinate X B $= y$ at any point represents a length of chain whose weight is equal to the tension at that point.

IV. Problem.—*Given two points in a catenary, and the length of chain between them; required the remainder of the curve.*

Let k be the horizontal distance between the two points, v their difference of level, l the length of chain between them. Those three quantities are the data.

The unknown quantities may be expressed in the following manner:—Let x_1, y_1, be the co-ordinates of the higher given point, and s_1 the arc terminating at it, all measured from the yet unknown vertex of the catenary, and x_2, y_2, s_2, the corresponding quantities for the lower given point.

Then the parameter m is to be found by a series of approximations from the following equation:—

$$m\left(e^{\frac{k}{2m}} - e^{-\frac{k}{2m}}\right) = \sqrt{l^2 - v^2};\ \ldots\ldots\ldots\ldots\ (3.)$$

the position of the vertex horizontally, by either of the equations,

$$x_1 = \frac{1}{2}\left(m \cdot \text{hyp. log.} \frac{l+v}{l-v} + k\right);\ x_2 = \frac{1}{2}\left(m \cdot \text{hyp. log.} \frac{l+v}{l-v} - k\right);\ (4.)$$

and the position of the vertex vertically by calculating y from x and m for either of the given points.

The part of a catenary in the neighbourhood of the vertex differs but little in figure from a parabola whose focal distance is $m \div 2$, half the modulus of the catenary; and in calculations for practical purposes within certain limits, the parabola may be used instead of the true catenary, its equation being more simple.

To show the amount of the difference between those curves, the following comparison is given, in which, instead of the finite equation of the catenary, an infinite converging series is substituted. The ordinate is supposed to be measured from the point O in fig. 113, at the distance m below the vertex.

Ordinate of the
- Catenary; $y = m\left(1 + \frac{x^2}{2\,m^2} + \frac{x^4}{24\,m^4} + \frac{x^6}{720\,m^6} + \&c.\right)$;
- Parabola; $y = m\left(1 + \frac{x^2}{2\,m^2}\right)$;

Slope of the
Catenary; $\frac{dy}{dx} = \frac{x}{m}\left(1 + \frac{x^2}{6m^2} + \frac{x^4}{120m^4} + \&c.\right)$;
Parabola; $\frac{dy}{dx} = \frac{x}{m}$;

Area of the
Catenary; $\int y\,dx = mx\left(1 + \frac{x^2}{6m^2} + \frac{x^4}{120m^4} + \&c.\right)$;
Parabola; $\int y\,dx = mx\left(1 + \frac{x^2}{6m^2}\right)$;

Length of the
Catenary; $s = x\left(1 + \frac{x^2}{6m^2} + \frac{x^4}{120m^4} + \&c.\right)$;
Parabola; $s = x\left(1 + \frac{x^2}{6m^2} - \frac{x^4}{40m^4} + \&c.\right)$.

It is to be borne in mind that the quantity denoted by m in these formulæ is *double* of that denoted by m in Article 125.

The following table exemplifies their results for the case $x = m \div 3$:—

	Ordinate $= m \times$	Slope.	Area $= mx \times$	Length $= x \times$
Catenary,.....	1·0561	0·3395	1·0186	1·0186
Parabola, ...	1·0556	0·3333	1·0185	1·0182
Difference,...	0·0005	0·0062	0·0001	0·0004

The **Catenary of Uniform Strength** is the figure assumed by a chain loaded in any manner, whose sectional area at each point is proportional to the tension. The figure assumed by such a chain, when loaded with its own weight only, was investigated by Mr. Davies Gilbert, in a paper published in the *Philosophical Transactions* for 1826. The Reverend Canon Moseley, in his *Mechanics of Engineering and Architecture*, has investigated the figure of the catenary of equal strength when the chain is loaded with suspending rods and a platform, as well as with its own weight. The resulting equations are of great complexity when in their exact form; but Mr. Moseley shows that in those cases which occur in practice the parabola forms a close approximation to the true curve, as it does in the case of the common catenary.

Under the head of "Metallic Structures," it will be shown how far it is useful in practice to take into account the peculiarities of the catenary of uniform strength.

129. **Centre of Gravity of a Flexible Structure.** (*A. M.*, 176.)—In every case in which a perfectly flexible structure, such as a cord, a chain, or a funicular polygon, is loaded with weights only, the figure of stable equilibrium in the structure is that which corresponds to the lowest possible position of the centre of gravity of the entire load. This principle enables all problems respecting the equilibrium of vertically loaded flexible structures to be solved by means of the "Calculus of Variations;" but it has not hitherto been much applied to practical questions.

130. **Transformation of Frames and Chains.** (*A.M.*, 166.)—The principle explained in Article 101, p. 150, of the transformation of a set of lines representing one balanced system of forces into another set of lines representing another system of forces which is also balanced, by means of what is called "PARALLEL PROJECTION," being applied to the theory of frames, takes the following form:—

If a frame whose lines of resistance constitute a given figure, be balanced under a system of external forces represented by a given system of lines, then will a frame whose lines of resistance constitute a figure which is a parallel projection of the original figure, be balanced under a system of forces represented by the corresponding parallel projection of the given system of lines; and the lines representing the stresses along the bars of the new frame will be the corresponding parallel projections of the lines representing the stresses along the bars of the original frame.

This theorem enables the conditions of equilibrium of any unsymmetrical frame which happens to be a parallel projection of a symmetrical frame, to be deduced from the conditions of equilibrium of the symmetrical frame.

The principle of transformation by parallel projection is applicable to continuously loaded chains as well as to polygonal frames. For instance, the bridge-chain with sloping rods of Article 126, p. 191, might be treated as a parallel projection of a bridge-chain with vertical rods, made by substituting oblique for rectangular co-ordinates.

The algebraical expressions for the alterations made by parallel projection in the co-ordinates of a loaded chain or cord, and in the forces applied to it, are as follows:—

In the original figure, let y be the vertical co-ordinate of any point, and x the horizontal co-ordinate. Let P be the vertical load applied between any point B of the chain and its lowest point A; let $p = \frac{dP}{dx}$ be its intensity per horizontal unit of length; let H be the horizontal component of the tension; let R be the tension at the point B.

Suppose that in the transformed figure, the vertical ordinate y',

and the vertical load P′, which is represented by a vertical line, are unchanged in length and direction, so that we have

$$y' = y;\ P' = P;\ \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots(1.)$$

but for each horizontal co-ordinate x, let there be substituted a horizontal or oblique co-ordinate x', inclined at the angle j to the horizon (which may be $= 0$), and altered in length by the constant ratio $\frac{x'}{x} = a$. Then for the horizontal tension H, there will be substituted a horizontal or oblique tension H′, parallel to x', and altered in the same proportion with that co-ordinate; that is to say,

$$x' = a\,x;\ H' = a\,H\ \ldots\ldots\ldots\ldots\ldots(2.)$$

The original tension at B is the resultant of the vertical load P and the horizontal tension H. Let R be its amount, and i its inclination to H; then

$$R = \sqrt{P^2 + H^2};\ \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots(3.)$$

and the ratios of those three forces are expressed by the proportion

$$P : H : R :: \tan i : 1 : \sec i :: \sin i : \cos i : 1 \ldots\ldots(4.)$$

Let R′ be the amount of the tension at the point B in the new structure, corresponding to B, and let i' be its inclination to the horizontal or oblique co-ordinate x'; then

$$R' = \sqrt{(P'^2 + H'^2 \pm 2\,P'\,H' \sin j)} \ldots\ldots\ldots(5.)$$

$$P' : H' : R' :: \sin i' : \cos (i' \pm j) : \cos j \ldots\ldots\ldots(6.)$$

The alternative signs $\pm$ are to be used according as i' and j $\left\{\begin{matrix}\text{agree}\\ \text{differ}\end{matrix}\right\}$ in direction.

The *intensity* of the load in the transformed structure *per unit of length* measured along $d\,x'$, whether horizontally or obliquely, is

$$p' = \frac{d\,P'}{d\,x'} = \frac{p}{a};\ \ldots\ldots\ldots\ldots\ldots\ldots\ldots(7.)$$

and if x' be oblique, and the intensity of the load be estimated *per unit of horizontal length*, it becomes

$$p' \sec j = \frac{p}{a \cdot \cos j}\ \ldots\ldots\ldots\ldots\ldots\ldots\ldots(8.)$$

131. The **Transformed Catenary** furnishes a good example of the transformations of chains, being derived by parallel projection from the common catenary. It has already been stated (see Article 128, equations 1) that in the common catenary the area O A B X,

fig. 113, is proportional to the arc A B, being equal to a rectangle whose sides are respectively the modulus $m = $ O A, and a straight line equal to the arc A B. Hence the common catenary is the curve of equilibrium for a chain supporting a load which, whether arising from its own weight alone or from other weights also, is proportional upon any given arc A B of the chain, to the area enclosed between that arc, the two ordinates A O and B X, and the *directrix* O X, which is at the depth m below the vertex; the *intensity* of the load at any point B being proportional to the ordinate $y = $ B X. This condition of the chain may be represented to the mind by conceiving the *whole* load to consist of the weight of an uniformly thick sheet of some uniformly heavy substance, bounded above by the catenary and below by the straight line O X. Let w denote the weight of an unit of area (say a square foot) of that sheet; then in the *Common Catenary*,

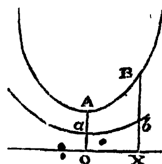


Fig. 113.

$$\left.\begin{array}{l}\text{the horizontal tension } \mathrm{H} = w \cdot \mathrm{O\,A}^2 = w m^2; \\ \text{the intensity of the load at B} = p = w y = \frac{w m}{2}\left(e^{\frac{x}{m}} + e^{-\frac{x}{m}}\right); \\ \text{the load between A and B} = \mathrm{P} = w \cdot \mathrm{O\,A\,X\,B} = w \int y \, d x \\ \qquad = \frac{w m^2}{2}\left(e^{\frac{x}{m}} - e^{-\frac{x}{m}}\right); \\ \text{the tension at B} = \sqrt{\mathrm{P}^2 + \mathrm{H}^2} = w\, m\, y = \frac{w m^2}{2}\left(e^{\frac{x}{m}} + e^{-\frac{x}{m}}\right). \end{array}\right\} (1.)$$

Now suppose a curve to be made, such as is represented by $a\,b$ in fig. 113, by preserving the horizontal abscissa of each point in the chain, but altering its vertical ordinate in a constant ratio; so that

$$\left.\begin{array}{c}\mathrm{O\,A} : \mathrm{O}\,a : : \mathrm{X\,B} : \mathrm{X}\,b; \\ \text{or denoting O } a \text{ by } y_0, \text{ and X } b \text{ by } y' \\ m : y_0 : : y : y'; \end{array}\right\} \ldots\ldots (2.)$$

Then the new curve, or TRANSFORMED CATENARY, $a\,b$, is the form of equilibrium for a chain so loaded that the load on any arc $a\,b$ is proportional to the area O $a\,b$ X, and the intensity at the point b to the ordinate X b. In the transformed catenary all the horizontal forces remain the same as in the original catenary; while all the vertical forces are altered in the ratio $y_0 : m$; that is to say,

$$\left.\begin{aligned}
&\text{The horizontal tension } H' = H = w\, m^2;\\
&\text{the intensity of the load at } b = p' = w\, y' = \frac{w\, y_0}{2}\left(e^{\frac{x}{m}} + e^{-\frac{x}{m}}\right);\\
&\text{the load between } a \text{ and } b = P' = w \int y'\, d\, x\\
&\qquad = \frac{w\, m\, y_0}{2}\left(e^{\frac{x}{m}} - e^{-\frac{x}{m}}\right);\\
&\text{the tension at } b = \sqrt{P'^2 + H^2}.
\end{aligned}\right\} \quad (3.)$$

In the course of the application of these principles, the following problem may occur:- *given, the directrix* O X, *the vertex* a *of the chain, and a point of support* b; *it is required to complete the figure of the chain.* For this purpose it is necessary and sufficient to find the modulus m, which is done by means of the following formula; let $y_0 = \text{O}\, a$ be the ordinate at a, y' the ordinate at the point of support, x the horizontal distance O X; then,

$$m = \frac{x}{\text{hyp. log.} \left(\frac{y'}{y_0} + \sqrt{\frac{y'^2}{y_0^2} - 1}\right)} \ldots\ldots\ldots\ldots (4.)$$

The principal use of the transformed catenary is as a figure for arches. (See the next Article.)

132. **Linear Arches or Ribs in general—Their Transformation.** (*A. M.*, 178.)—Conceive a cord or chain to be exactly inverted, so that the load applied to it, unchanged in direction, amount, and distribution, shall act inwards instead of outwards; suppose, further, that the cord or chain is in some manner stayed or stiffened, so as to enable it to preserve its figure and to resist a thrust; it then becomes a *linear arch* or *equilibrated rib;* and for the pull at each point of the original chain is now substituted an exactly equal *thrust* along the rib at the corresponding point.

Linear arches do not actually exist; but the propositions respecting them are applicable to the lines of resistance of real arches and arched ribs, in a manner which will be explained in treating of masonry.

All the propositions and equations of the preceding Articles, respecting cords or chains, are applicable to linear arches, substituting only a *thrust* for a *pull*, as the stress along the line of resistance.

The principles of Article 123, p. 185, are applicable to linear arches in general, with external forces applied in any direction.

The principles of Articles 124, 125, 128, 130, and 131, pp. 187 to

202, are applicable to linear arches under *vertical loads*; and in such arches, the quantity denoted by H in the formulæ represents a *constant thrust*, in a direction perpendicular to that of the load.

The form of equilibrium for a linear arch under an uniform load is a *parabola*, similar to that described in Article 125, p. 188.

In the case of a linear arch under a vertical load, the word *intrados* is used to denote the figure of the arch itself, and the word *extrados*, to denote a line traversing the *upper* ends of ordinates, drawn *upwards* from the intrados, of lengths proportional to the intensities of the load.

The figure of equilibrium for a linear arch with a horizontal extrados is either a catenary or a transformed catenary inverted; and the equations of Article 131 are applicable to the determination of its figure and of the forces which act in it, w being taken to denote the weight of so much of the loading material as is contained in one square foot of the area between the extrados O X, fig. 113, and the intrados A B or a b. This is what is called by most mechanical writers, an "equilibrated arch."

The principles of Article 130, relative to the transformation of cords and chains, are applicable also to linear arches or ribs. This subject will be further considered in the sequel.

133. **Circular Rib for Fluid Pressure.** (*A. M.*, 179.)—A linear arch, to resist an uniform normal pressure from without, should be circular.

In fig. 114, let A B A B be a circular linear arch, rib, or ring,

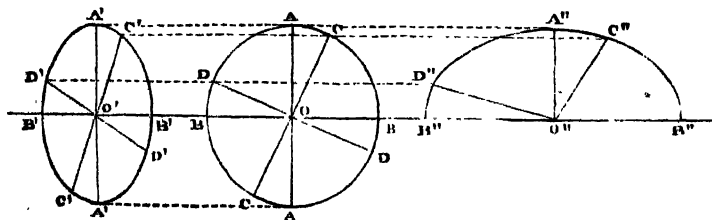


Fig. 114.

whose centre is O, pressed upon from without by a normal pressure of uniform intensity.

In order that the intensity of that pressure may be conveniently expressed in units of force per unit of area, conceive the ring in question to represent a vertical section of a cylindrical shell, whose length, in a direction perpendicular to the plane of the figure, is one foot. Let p denote the intensity of the external pressure, in lbs. on the square foot; r the radius of the ring in feet; T the

thrust exerted round it, which, because its length is one foot, is a thrust in lbs. per foot of length of the cylinder; then,

$$\therefore \quad T = p\, r \quad \text{(1.)}$$

that is to say:—*the thrust round a circular ring under an uniform normal pressure is the product of the pressure on an unit of circumference by the radius.*

The uniform normal pressure p, if not actually caused by the thrust of a fluid, is similar to fluid pressure; and it is equivalent to a pair of conjugate pressures in any two directions at right angles to each other of equal intensity. For example, let x be vertical, y horizontal, and let p_x, p_y, be the intensities of the vertical and horizontal pressure respectively; then

$$p_x = p_y = p; \quad \text{(2.)}$$

and the same is true for any pair of rectangular pressures; and if P be the total vertical pressure, and H the total horizontal pressure, exerted upon one quadrant A B of the circle, we have

$$H = P = T = p\, r \quad \text{(3.)}$$

134. **Elliptical Ribs for Uniform Pressures.** (*A. M.*, 180.)—If a linear arch has to sustain the pressure of a mass in which the pair of conjugate thrusts at each point are uniform in amount and direction, but not equal to each other, all the forces acting parallel to any given direction will be altered from those which act in a fluid mass, by a given constant ratio; so that they may be represented by *parallel projections* of the lines which represent the forces that act in a fluid mass. Hence the figure of a linear arch, which sustains such a system of pressures as that now considered, must be a parallel projection of a circle; that is, an *ellipse.* To investigate the relations which must exist amongst the dimensions of an elliptic linear arch under a pair of conjugate pressures of uniform intensity, let A′ B′ A′ B′, B″ A″ B″, in fig. 114, represent elliptic ribs, transformed from the circular rib A B A B by parallel projection, the vertical dimensions being unchanged, and the horizontal dimensions either expanded (as B″ B″), or contracted (as B′ B′), in a given uniform ratio denoted by c; so that r shall be the vertical and $c\, r$ the horizontal semi-axis of the ellipse; and if x, y, be respectively the vertical and horizontal co-ordinates of any point in the circle, and x', y', those of the corresponding point in the ellipse, we shall have

$$x' = x; \quad y' = c\, y \quad \text{(1.)}$$

If C C, D D, be any pair of diameters of the circle at right angles to each other, their projections will be a pair of conjugate diameters of the ellipse, as C′ C′, D′ D′; that is, diameters each of which is parallel to a tangent at the end of the other.

Let P′ be the total vertical pressure, and H′ the total horizontal pressure, on one quadrant of the ellipse, as A′ B′, or A″ B″; P′ is also the vertical thrust on the rib at B′ or B″, and H′ the horizontal thrust at A′ or A″.

Then, by the principle of transformation,

$$\left.\begin{array}{r} P' = P = T = p\,r; \\ H' = c\,H = c\,T = c\,p\,r; \end{array}\right\} \qquad (2.)$$

or, ***the total thrusts are as the axes to which they are parallel,***

Further, let P″ be the total pressure, parallel to any semidiameter of the ellipse (as O′ D′ or O″ D″) on the quadrant D′ C′ or D″ C″, which force is also the thrust of the rib at C′ or C″, the extremity of the diameter conjugate to O′ D′ or O″ D″; and let O′ D′ or O″ D″ $= r'$; then

$$P = p\,r'; \qquad (3.)$$

or, ***the total thrusts are as the semidiameters to which they are parallel.***

Next, let p'_x, p'_y, be the ***intensities*** of the conjugate vertical and horizontal pressures on the elliptic arch; that is, of the "***principal stresses.***" (Articles 109, 112.) Each of those intensities being found by dividing the corresponding total pressure by the area of the plane to which it is normal, they are given by the following equation:—

$$p'_x = \frac{P'}{[illegible]} = \frac{p}{[illegible]} \quad p'_y = \frac{H'}{[illegible]} \quad c\,p \qquad (4.)$$

so that ***the intensities of the principal pressures are as the squares of the axes of the elliptic rib to which they are parallel.***

Hence, to adapt an elliptic rib to uniform vertical and horizontal pressures, ***the ratio of the axes of the rib must be the square root of the ratio of the intensities of the principal pressures;*** that is,

$$\frac{O\,B'}{O\,A'} = c = \sqrt{\frac{p'_y}{p'_x}}. \qquad (5.)$$

The external pressure on any point D′ or D″, of the elliptic rib is directed towards the centre, O′ or O″, and its intensity, per unit of area of the plane to which it is conjugate (O′ C′ or O″ C″), is given by the following equation, in which r' denotes the semidiameter (O′ D′ or O″ D″) parallel to the pressure in question, and r'' the conjugate semidiameter (O′ C′ or O″ C″):—

$$p' = \frac{P''}{r''} = p. \qquad (6.)$$

that is, *the intensity of the pressure in the direction of a given diameter is directly as that diameter and inversely as the conjugate diameter.*

Let p'' be the intensity of the external pressure in the direction of the semidiameter r''. Then it is evident that

$$p' : p' \quad \ldots (7.)$$

that is, *the intensities of a pair of conjugate pressures are to each other as the squares of the conjugate diameters of the elliptic rib to which they are respectively parallel.*

135. **Distorted Elliptic Rib.** (*A. M.*, 181.)—To adapt an elliptic rib to the sustaining of the pressure of a mass in which, while the state of stress is uniform, the pressure conjugate to a vertical pressure is not horizontal, but inclined at a given angle j to the horizon, the figure of the ellipse must be derived from that of a circle by the substitution of inclined for horizontal co-ordinates.

In fig. 115, let B A C be a semicircular arch on which the external pressures are normal and uniform, and of the intensity p, as before; the radius being r, and the thrust round the arch, and load on a quadrant, being as before, $P = H = T = p\,r$. Let D be any point in the circle, whose co-ordinates are vertical, $\text{O E} = x$, horizontal, $\text{E D} = y$. Let B′ A′ C′ be a semi-elliptic arch, in which the vertical ordinates are the same with those of the circle, while for

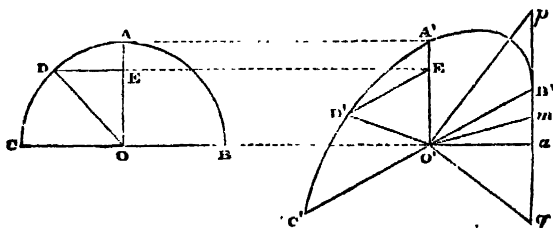


Fig. 115.

each horizontal ordinate is substituted an ordinate inclined to the horizon by the constant angle j, and bearing to the corresponding horizontal ordinate of the circle the contant ratio c; that is to say, let

$$\overline{\text{O}'\text{E}'} = x' = x;\ \overline{\text{E}'\text{D}'} = y' = c\,y;\ \angle \text{E E}'\text{D}' = j \ldots (1.)$$

Then for the vertical semidiameter of the circle $\text{O A} = r$, will be substituted the equal vertical semidiameter of the ellipse $\text{O}'\text{ A}' = r$; and for the horizontal semidiameter of the circle $\text{O B} = r$, will

substituted the inclined semidiameter of the ellipse $O'B' = c\,r$, which is *conjugate* to the vertical semidiameter.

The forces applied to the elliptic arch are to be resolved into vertical and *inclined* components, parallel to O′ A′ and C′ B′, instead of vertical and horizontal components. Let P′ denote the total vertical pressure, and H′ the total inclined pressure, on either of the elliptic quadrants, C′ A′, A′ B′; H′ is also the inclined thrust of the arch at A′, and P′ the vertical thrust at B′ or C′. Then

$$\left.\begin{array}{l} P' = P = p\,r; \\ H' = c\,H = c\,P = c\,p\,r; \end{array}\right\} \dots\dots\dots\dots\dots\dots (2.)$$

that is to say, those forces are, as before, *proportional to the diameters to which they are parallel.*

Let p'_x be the intensity of the vertical pressure on the elliptic arch per unit of area of the inclined plane to which it is conjugate, C′ B′; let p'_y, be the intensity of the inclined pressure per unit of area of the vertical plane to which it is conjugate; then

$$p'_x = \frac{P'}{c\,r} = \frac{p'}{c}\,;\ p_y = \frac{H'}{r} = c\,p;\ c = \sqrt{\frac{p'_y}{p'_x}};\dots\dots\dots (3).$$

so that, as before, *the intensities of the conjugate pressures are as the squares of the diameters to which they are parallel.*

The thrust of the arch at any point D′ is as before, proportional to the diameter conjugate to O′ D′.

It is sometimes convenient to express the intensity of the vertical pressure per unit of area of the *horizontal projection* of the space over which it is distributed; this is given by the equation

$$p'_x \cdot \sec j = \frac{p}{c \cdot \cos j};\ \dots\dots\dots\dots\dots\dots\dots\dots (4.)$$

It is to be borne in mind that this is not the pressure on unity of area of a horizontal plane (which pressure is inversely as the horizontal diameter of the ellipse, and directly as the diameter conjugate to that diameter, to which latter diameter it is parallel), but the pressure on that area of a plane inclined at the angle j, whose horizontal projection is unity.

The following geometrical construction serves to determine the major and minor axes of the ellipse B′ A′ C′.

Draw $O'a \perp$ and $=$ O′ A′; join B′ a, which bisect in m; in B′ a produced both ways take $m\,p = m\,q = O'\,m$; join $O'\,p$, $O'\,q$; these lines, which are perpendicular to each other, are the *directions* of the axes of the ellipse, and the *lengths* of those axes are respectively equal to the segments of the line $p\,q$, viz., $\overline{B'p} = \overline{a\,q}$, $\overline{B'q} = \overline{a\,p}$.

The following is the algebraical expression of this solution:—Let A denote the major and B the minor semi-axis of the ellipse.

Then

$$\left.\begin{array}{l} A = \frac{r}{2} \left\{ \sqrt{(1 + c^2 + 2\,c \cdot \cos j)} + \sqrt{(1 + c^2 - 2\,c \cdot \cos j)} \right\}; \\ B = \frac{r}{2} \left\{ \sqrt{(1 + c^2 + 2\,c \cdot \cos j)} - \sqrt{(1 + c^2 - 2\,c \cdot \cos j)} \right\}; \end{array}\right\} \quad (5.)$$

The angle $\angle$ B′ O′ p, which the nearest axis makes with the diameter C′ B′, is found by the equation

$$\sin B' O' p = \frac{B}{c\,r} \sqrt{\left(\frac{A^2 - c^2 r^2}{A^2 - B^2}\right)} \text{ or } \frac{A}{c\,r} \sqrt{\left(\frac{B^2 - c^2 r^2}{A^2 - B^2}\right)}; \quad (6.)$$

according as that axis is the longer —— the shorter.

136. **Ribs for Normal Pressure—Hydrostatic Arch.** (*A. M.*, 182, 183, 319 A.)—The condition of a linear arch of any figure at any point where the pressure is normal, is similar to that of a circular rib of the same curvature under a pressure of the same intensity; and hence the following principle:—*the thrust at any normally pressed point of a rib is the product of the radius of curvature by the intensity of the pressure;* that is, denoting the radius of curvature by ρ, the normal pressure per unit of length of curve by p, and the thrust by T,

$$T = p\,\rho \quad (1.)$$

It is further evident, that *if the pressure be normal at every point of the rib*, the thrust must be constant at every point; for it can vary only by the application of a tangential pressure to the arch; and *the radius of curvature must be inversely as the pressure.*

This is the case in the HYDROSTATIC ARCH, which is a linear arch or rib suited for sustaining normal pressure at each point proportional, like that of a liquid in repose, to the depth below a given horizontal plane.

The radius of curvature at a given point in the hydrostatic arch being inversely proportional to the intensity of the pressure, is also inversely proportional to the depth below the horizontal plane at which vertical ordinates representing that intensity commence.

In fig. 116, let Y O Y represent the level surface from which the pressure increases at an uniform rate downwards, so as to be similar to the pressure of a liquid having its upper surface at Y O Y. Let A be the crown of the hydrostatic arch, being the point where it is nearest the level surface, and consequently horizontal. Let co-ordinates be measured from the point O in the level surface, directly above the crown of the arch; so that O X = Y C = x shall be the

vertical ordinate, and O Y = X C = y the horizontal ordinate, of any point, C, in the arch. Let O A, the least depth of the arch below

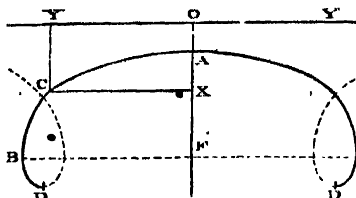


Fig. 116.

the level surface, be denoted by x_0, the radius of curvature at the crown by ϱ_0, and the radius of curvature at any point, C, by ϱ.

Let w be the weight of an unit of volume of the liquid, to whose pressure the load on the arch is equivalent. Then the intensities of the external normal pressure at the crown A, and at any point, C, are expressed respectively by

$$p_0 = w\, x_0;\ p = w\, x. \ \ldots\ldots\ldots\ldots\ldots\ldots\ldots (2.)$$

The thrust along the rib, which is a constant quantity, is given by the equation

$$\mathrm{T} = p_0\, \varrho_0 = w\, x_0\, \varrho_0 = p\, \varrho = w\, x\, \varrho; \ \ldots\ldots\ldots\ldots (3.)$$

from which follows the following geometrical equation, being that which characterizes the figure of the arch:—

$$x\, \varrho = x_0\, \varrho_0. \ \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (3.)$$

When x_0 and ϱ_0 are given, the property of having the radius of curvature inversely proportional to the vertical ordinate from a given horizontal axis enables the curve to be drawn approximately, by the junction of a number of short circular arcs, as in fig. 117; the radius of each short arc being inversely as the mean depth of that arc below O Y. The curve is found to present some resemblance to a trochoid (with which, however, it is by no means identical). At a certain point, B, it becomes vertical, beyond which it continues to turn, until at D, it becomes horizontal; at this point its depth below the level surface is greatest, and its radius of curvature least. Then ascending, it forms a loop, crosses its former course, and proceeds towards E to form a second arch similar

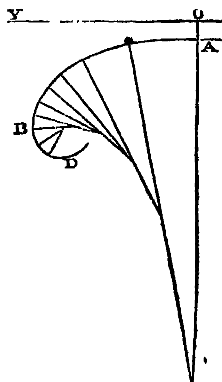


Fig. 117.

to B A B. Its coils, consisting of alternate arches and loops, all similar, follow each other in an endless series.

It is obvious that only one coil or division of this curve, viz., from one of the lowest points, D, through a vertex, A, to a second point, D, is available for the figure of an arch; and that the portion B A B, above the points where the curve is vertical, is alone available for supporting a load.

Let x_1, y_1, be the co-ordinates of the point B. The vertical load above the semi-arch A B is represented by

$$P_1 = w \int_0^{y_1} x \, d y ; \text{.......................} (4.)$$

and this being sustained by the thrust T of the arch at B, must be equal to that thrust; whence follows the equation

$$x \, \varrho = x_0 \, \varrho_0 = \int_0^{y_1} x \, d y. \text{ ....} (5.)$$

The vertical load above any point C, is

$$P = w \int_0^{y} x \, d y = T \sin i ; \text{ ....} (6.)$$

i being the inclination of the arch to the horizon.

The horizontal external pressure against the semi-arch from B to A is the same with that on a vertical plane, A F, immersed in a liquid of the specific gravity w, with its upper edge at the depth x_0 below the surface (see Article 107, p. 166); and it is balanced by the thrust T at the crown of the arch, so that its amount is

$$H = w \int_{x_0}^{x_1} p \, d x = w \cdot \frac{x_1^2 - x_0^2}{2} = T = p \, \varrho. \text{........} (7.)$$

Equation 7 gives for the value of the vertical tangent ordinate at B,

$$x_1 = \sqrt{x_0^2 + 2 x_0 \, \varrho_0}. \text{ ....} (8.)$$

The horizontal external pressure between B and any point, C, is equal to the pressure of a liquid of the specific gravity w on a vertical plane X F with its upper edge immersed to the depth x, so that its amount is

$$w \int_x^{x_1} p \, d x = w \cdot \frac{x_1^2 - x^2}{2} = T \cos i. \text{ ....} (9.)$$

The various geometrical properties of the figure of the hydro-

static arch expressed by the preceding equations are thus summed up in one formula,—

$$x_0 \rho_0 = x \rho = \int_0^{y_1} x\,d\,y = \frac{\int_0^y x\,d\,y}{\sin i} = \frac{x_1^2 - x_0^2}{2} = \frac{x_1^2 - x^2}{2 \cos i} \quad (10.)$$

To obtain exact expressions for the horizontal co-ordinate y, whose maximum value is the half-span y_1, and also for the lengths of arcs of the curve, it is necessary to use elliptic functions. Those functions are so little studied that their use will not be further adverted to here. The reader is therefore referred, for further information on that point, to the papers of M. Yvon-Villarceaux, in the *Mémoires des Savans étrangers*, vol. xii., and in the *Revue de l'Architecture* for 1845, and to *A Manual of Applied Mechanics*, p. 193.

For practical purposes, the following approximation is in general sufficient:—

PROBLEM.—Given the *rise* F A $= a$ and *half-span* F B $= y_1$, of a proposed hydrostatic arch: it is required to find the depth of load x_0 at the crown, and the radii of curvature, ρ_0, ρ_1, at the crown A and springing B, to draw the arch, and to compute its load and thrust.

A close approximation to x_0 is given as follows:—

$$\left.\begin{array}{l} \text{Let } b = y_1 + \dfrac{y_1^2}{30\,a}; \text{ then} \\ x_0 = a \cdot \dfrac{a^3}{b^3 - a^3}. \end{array}\right\} \ldots\ldots\ldots\ldots(11.)$$

Then observing that $x_1 = $ O F $= x_0 + a$, we find, from equation 10,

$$\left.\begin{array}{l} \rho_0 = \dfrac{x_1^2 - x_0^2}{2\,x_0} = a + \dfrac{a^2}{2\,x_0}; \\ \rho_1 = \dfrac{x_1^2 - x_0^2}{2\,x_1} = a - \dfrac{a^2}{2\,(x_0 + a)}. \end{array}\right\} \ldots\ldots\ldots(12.)$$

These radii being known, the figure of the arch can be drawn approximately by small circular arcs, as in fig. 117, already described.

The load on the half-arch, and the thrust, which are equal to each other, are now to be computed by equation 7, p. 210.

A mechanical mode of drawing a hydrostatic arch is based on the fact, that its figure is identical with one of the "elastic

curves" or forms assumed by an uniformly stiff spring when bent. (*A. M.*, 319A.)

The accuracy of figure and uniformity of stiffness of a spring are to be ascertained by the two following tests:—

First, the spring when unstrained should be exactly straight:

Secondly, when bent into a hoop by pinching the two ends together, it should form an exact circle.

A spring A (fig. 118), fulfilling these conditions, is to have its ends fixed to two bars at B and D, and the other ends of those bars, C and E, are to be pulled directly asunder. Then the straight line C E in which the forces so pulling the bars are exerted, will represent the upper surface of the loading material, and the spring A will assume the figure of the corresponding hydrostatic arch. Any proportion of rise to span can be obtained by varying the tension on the ends of the bars, and the proportion which their lengths bear to the length of the spring.

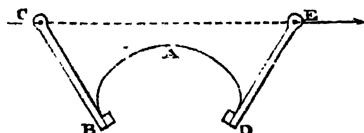


Fig. 118.

137. **Transformed Hydrostatic, or Geostatic Arch.** (*A. M.*, 184.)— It has been proposed, by this term, to denote a linear arch of a figure suited to sustain a pressure similar to that of earth, which (as will be shown in the sequel) consists, in a given vertical plane, of a pair of conjugate pressures, one vertical and proportional to the depth below a given plane, horizontal or sloping, and the other parallel to the horizontal or sloping plane, and bearing to the vertical pressure a certain constant ratio, depending on the nature of the material, and other circumstances to be explained in the sequel. In what follows, the horizontal or sloping plane will be called the *conjugate plane*, and ordinates parallel to its line of steepest declivity, when it slopes, or to any line in it, when it is horizontal, *conjugate ordinates*. The intensity of the vertical pressure will be estimated per unit of area of the *conjugate plane;* and the pressure parallel to the line of steepest declivity of that plane, when it slopes, or to any line in it, when it is horizontal, will be called the *conjugate pressure*, and its intensity will be estimated per unit of area of a vertical plane.

Let p_x denote the intensity of the vertical pressure, and p_y that of the conjugate pressure, at any given point. Construct the figure of a hydrostatic arch suited to sustain fluid pressure of the intensity p_x. Then the transformed arch is to be drawn by preserving all the vertical co-ordinates of the hydrostatic arch, and changing the horizontal co-ordinates into conjugate

co-ordinates, having their lengths altered in the constant ratio.

$$c = \sqrt{\frac{p_y}{p_x}}; \qquad (1.)$$

exactly as in the case of transforming a circular into an elliptic rib, in Article 134 and 135.

The radius of curvature at the springing is altered in the ratio $\frac{1}{c}$, and that at the crown in the ratio $c^2 : 1$.

Let P, P′, be the total vertical loads on one-half of the original and transformed half-arch respectively; H = P and H′, their respective conjugate thrusts, of which the former is horizontal, and the latter may be horizontal, or inclined at the angle j.

Then the bulk of the transformed arch with its load is altered in the ratio of $c \cos j : 1$; and if the new and transformed arches be of the same material, we find,

$$P' = c \cos j \cdot P; \; H' = c\,P' = c^2 \cos j \cdot P. \qquad (2.)$$

138. A **Linear Arch or Rib of any Figure** (*A. M.*, 185, 187), under a *Vertical load distributed in any manner*, being given, it is always possible to determine a system of horizontal or sloping pressures, which, being applied to that rib, will keep it in equilibrio. These last may be called the *Conjugate Pressures*.

The only case which will here be given in detail is that in which the conjugate pressures are horizontal, and the load symmetrically distributed on each side of the crown of the arch, A, fig. 119.

PROBLEM I.—To find the total horizontal pressure against the rib below a given point. The following is the graphic solution of this problem:—Let C be any point in the rib.

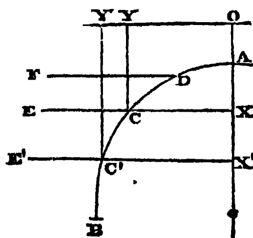


Fig. 119

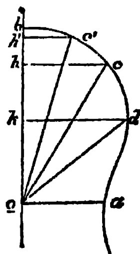


Fig. 120.

In the diagram of forces, fig. 120, draw $o\,c$ parallel to a tangent to the rib at C. Draw the vertical line $o\,b$ as a *scale of loads*, on which take $o\,h$ = P to represent the vertical load supported on the

arc A C. Through h draw the horizontal line $h\,c$, cutting $o\,c$ in c; then $o\,c = T$ will be the thrust along the rib at C, and $h\,c = H$, the horizontal component of that thrust, will be the *total horizontal pressure which must be exerted against* C B, *the part of the rib below* C.

This solution is expressed algebraically as follows:—As the origin of co-ordinates in fig. 119, take any convenient point O in the vertical line O A traversing the crown of the arched rib; let $O\,X = Y\,C = x$ and $O\,Y = X\,C = y$ be the co-ordinates of the point C; so that if i is the inclination of the arch at C (and of the line $o\,c$ in fig. 120) to the horizon, $d\,y \div d\,x = \text{cotan}\ i$.

Then,

$$H = P\frac{d\,y}{d\,x} = P\ \text{cotan}\ i;\ T = \sqrt{P^2 + H^2} = P\ \text{cosec}\ i. \quad (1.)$$

PROBLEM II.—To find the thrust at the crown of the rib.

The preceding process fails to give any result for the crown of the rib A; but the principle of Article 133, p. 204, shows, that if p_0 be the value of $d\,P \div d\,y$, the intensity of the load, at that point, the horizontal thrust is

$$T_0 = p_0\,\rho_0; \quad \ldots\ldots(2.)$$

ρ_0 being the radius of curvature of the rib at its crown.

PROBLEM III.—*To find the mean intensity of the horizontal pressure required in a given layer of the spandril;* that is, of the mass of material touching the convex side of the rib. (Fig. 119.)

Let C′ (fig. 119) be a point in the arch a short way below C, whose co-ordinates are $x + d\,x$, $y + d\,y$, so that $d\,x$ is the depth of the horizontal layer C E E′ C′. In the diagram of forces (fig. 120), draw $o\,c'$ parallel to a tangent to the rib at C′; on the vertical scale of loads take $o\,h' = P + d\,P$ to represent the vertical load on the arc A C′; draw the horizontal line $h'\,c'$ cutting $o\,c'$ in c'. Then $o\,c' = T'$ is the thrust along the rib at C′; and $h'\,c' = H'$, the horizontal component of that thrust, is the horizontal pressure which must be exerted against the part of the rib below C′; so that

$$h\,c - h'\,c' = H - H' = -\,d\,H, \quad \ldots\ldots(3.)$$

is the horizontal pressure to be exerted through the layer C E E′ C′ and

$$p_x = -\frac{d\,H}{d\,x} = -\frac{d}{d\,x}\left(P\frac{d\,y}{d\,x}\right) \quad \ldots\ldots(4.)$$

the intensity of that pressure.

The negative sign prefixed to d H denotes that if H *diminishes in going downwards*, as in the example given, pressure is required through the layer. Through those layers at which H *increases in going downwards*, either *tension from without*, or *pressure from within*, is required to keep the rib in equilibrio.

PROBLEM IV.—To find the greatest horizontal thrust, and the "point of rupture," and "angle of rupture."

First Solution.—By a graphic process. Through o in fig. 120, draw a number of radiating lines, such as $o\ c$, $o\ c'$, &c., parallel to the rib at various points, as C, C′, &c., and find as in Problem I. and III., the lengths of those lines so as to represent the thrust along the rib at the several points C, C′, &c. The length of the horizontal line $o\ a$, representing the thrust at the crown, is to be calculated as in Problem II. Through the points a, c, c', &c., thus found, draw a curve. Find the point d in that curve which is furthest from the scale of loads $o\ b$; then the horizontal line $d\ k = \mathrm{H}_0$ will represent the maximum horizontal thrust.

Join $o\ d$, and find the point D in fig. 119, at which the rib is parallel to $o\ d$; this is the "point of rupture," or point at which the horizontal thrust attains a maximum; and the "angle of rupture" is the inclination of the rib at that point, or $\angle\ d\ o\ a$ in fig. 120, which will be denoted in the sequel by i_0.

The horizontal plane D F is the upper boundary of that part of the spandril which exerts the maximum horizontal pressure H_0.

Second Solution.—By arithmetical trials. Compute, as in Problem I., the values of H for some points in the arch. Between the point which gives the greatest value of H in the first set of trials, and the two on either side of it, introduce intermediate points, for which compute the values of H, and repeat the process until the point of rupture is found with the desired degree of exactness.

Third Solution.—By the differential calculus and the solution of an equation. If the relations between x, y, and P can be expressed by equations, make the expression for the *intensity* of the horizontal pressure in equation 4 equal to 0; and by solving the equation so obtained, deduce the position of D and the values of i_0 and H_0. The equation to be solved has, in most cases, two roots, one of which corresponds to the crown of the arch A, and the other to the required point D; but it is easy to distinguish between them. If there are more than two roots, they indicate a set of points at each of which $p_y = 0$, and which are alternately points of $\left\{ \begin{matrix} \text{maximum} \\ \text{minimum} \end{matrix} \right\}$ horizontal thrust, according as $\frac{d^2\,\mathrm{H}}{d\,x^2}$ is $\left\{ \begin{matrix} \text{negative} \\ \text{positive} \end{matrix} \right\}$; that is, according as $\frac{d\,p_y}{d\,x}$ is $\left\{ \begin{matrix} \text{positive} \\ \text{negative} \end{matrix} \right\}$. Cases of this kind are of rare occurrence in practice.

If there is but one root, it corresponds to the crown of the rib; the hydrostatic arch (Article 136), is an example of this, in which the crown is the point of greatest horizontal thrust. In the Catenary (Article 128), and Transformed Catenary (Article 131), and other "curves of equilibrium" for vertical loads (Article 123), H is constant, and $p_y = 0$ for every point in the rib.

If the rib rises *vertically* from its springing-point, as at B, the whole of the horizontal pressure which sustains it is distributed through the layers of the spandril. (The term "*complete*" has lately been introduced to denote such a rib.) If the arch rises obliquely from such a springing-point as C, part at least of its greatest horizontal thrust consists of the horizontal component of the thrust along the rib at that point. Such a rib is said to be "*segmental.*"

Let i_1 denote the inclination to the horizon of the rib at its springing-point, and P_1 the whole vertical load from the crown to the springing-point: then the horizontal component in question is P_1 cotan i_1; so that $H_0 - P_1 \text{ cotan } i_1$ is the part of the greatest horizontal thrust which is distributed through the spandril.

PROBLEM V.—To find the position of the resultant of the maximum horizontal thrust.

From the point of rupture D down to the springing, conceive the spandril to be divided into horizontal layers. Let $d\,x$ denote the depth of any one of those layers; p_y the intensity of the horizontal pressure exerted by it against the rib;

x, the depth of its centre below O Y, fig. 119;
x_0, the depth of the joint of rupture below O Y;
x_1, the depth of the springing-point below O Y;

Then, x_H, the required depth of the resultant below O Y, may be expressed in either of the following forms:—

$$x_H = \frac{\int_0^{H_0} x\,d\,H}{H_0} = \frac{\int_{x_0}^{x_1} x\,p_y\,d\,x + x_1 P_1 \text{ cotan } i_1}{H_0} \quad \text{.....(5.)}$$

Example I.—In the *Catenary* and *Transformed Catenary*, and other ribs equilibrated under vertical loads,

$$H_0 = P_1 \text{ cotan } i_1;\ x_H = x_1. \quad \text{................(6.)}$$

Example II.—In a *Semicircular Rib* of the radius r under *uniform normal pressure* of the intensity p; let the origin of co-ordinates be at the crown of the arch.

$$H_0 = p\,r;\ x_{\text{H}} = \frac{r}{2}. \text{ ....(7.)}$$

As to semi-elliptic arches under conjugate uniform pressures, see Article 134, p. 205.

Example III.—In the *Hydrostatic Arch*, as in fig. 116, Article 136, p. 209, let the origin of co-ordinates be in the extrados above the crown; then

$$H_0 = w \cdot \frac{x_1^2 - x_0^2}{2};\ x_{\text{H}} = \frac{2}{3} \cdot \frac{x_1^3 - x_0^3}{x_1^2 - x_0^2}. \text{ ....(8).}$$

In the transformed hydrostatic or geostatic arch, x_{H} is the same as in the hydrostatic arch. As to the thrust, see Article 137, p. 213.

Example IV.—In a *Semicircular Rib* with a horizontal extrados, let r be the radius of the rib; let the origin of co-ordinates be at the *crown of the arch*; let $m\,r$ be the height of the extrados above the crown; and let w be the weight of each unit of vertical area of the load.

The intensity of the horizontal pressure through a given layer of the spandril is,

$$p_y = w\,r\left(1 + m - \cos i - \frac{i - \cos i \sin i}{2 \sin^3 i}\right) \text{..........(9.)}$$

The angle of rupture i_0 is found by solving the transcendental equation,

$$p_y = 0, \text{ ....(10.)}$$

This is to be done by successive approximations; and as a first approximation may be taken

$$i_0 = \text{arc} \cdot \cos \frac{1 + 3\,m}{2} \text{ approximately. ......(10 A.)}$$

The maximum thrust is given by the formula

$$H_0 = w\,r^2 \left\{(1 + m) \cos i_0 - \frac{\cos^2 i_0}{2} - \frac{i_0 \cot\text{an}\, i_0}{2}\right\};\ \text{(11.)}$$

and the depth of its resultant below the crown of the arch by the formula

$$x_{\text{H}} = \frac{r^2}{H_0} \cdot \int_{i_0}^{90°} p_y \sin i\,(1 - \cos i)\,d\,i \text{ ....(12.)}$$

Example V.—In a *Circular Segmental Rib* with a *horizontal extrados*, let i_1 be the inclination of the arch at the springing,

and P_1 the vertical load at the springing, and let the rest of the notation be as in the last example.

Let the angle of rupture be found as before.

• *Case* 1.—$i_0 > or = i_1$. Then

$$H_0 = P_1 \text{ cotan } i_1 ; \; x_{\text{H}} = r (1 - \cos i_1) \ldots\ldots\ldots (13.)$$

Case 2.—$i_0 < i_1$. Find H_0 and p_v as in Example IV.; then

$$x_{\text{H}} = \frac{1}{H_0} \left\{ r^2 \int_{i_0}^{i_1} p_v \sin i (1 - \cos i) \, d i + r P_1 \text{ cotan } i_1 (1 - \cos i_1) \right\} \quad (14.)$$

Example VI.—*Semi-elliptic Rib. with a horizontal extrados.* Conceive a semicircular rib whose radius is equal to the rise of the semi-elliptic rib, and extrados at the same height above the crown, and find H_0 and x_{H} for the semicircular rib as in Example IV. x_{H} for the semi-elliptic rib will be the same; and the thrust is to be found by the principle of transformation, as in Article 134, p. 205.

The best form, however, for oval complete ribs is that of the hydrostatic arch, which sufficiently resembles the semi-ellipse to be substituted for it.

139. **Pointed Rib.**—If a linear arch, as in fig. 121, consists of two arcs, B C, C B, meeting in a point at C, it is necessary to equilibrium that there should be concentrated at the point C a load equal to that which would have been distributed over the two arcs A C, C A, extending from the point C to the respective crowns, A, A, of the curves of which two portions form the pointed arch.

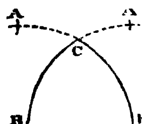


Fig. 121.

Under the head of "Masonry" it will be shown how and under what circumstances that concentration of load becomes unnecessary in stone arches.

140. **Stability of Blocks.** (*A. M.*, 205, 206.)—The conditions of stability of a single block supported upon another body at a plane joint may be thus summed up:—

In fig. 122, let A A represent the upper block, B B part of the supporting body, e E the joint, C its centre of pressure, P C the resultant of the whole pressure distributed over the joint, N C, T C, its components perpendicular and parallel to the joints respectively. Then the conditions of stability are the following:—

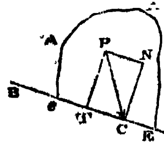


Fig. 122.

1. *In order that the block may not slide, the*

obliquity of the pressure must not exceed the angle of repose (see Article 110, p. 171), that is to say,

$$\angle \text{P C N} \leqq \angle \Phi \ldots\ldots\ldots\ldots\ldots\ldots (\text{I.})$$

II. *In order that the block may be in no danger of overturning, the ratio which the deviation of the centre of pressure from the centre of figure of the joint bears to the length of the diameter of the joint traversing those two centres, must not exceed a certain fraction.* The value of that fraction varies, according to circumstances, which will be explained in treating of Masonry, from one-eighth to three-eighths.

The first of these conditions is called that of *stability of friction*, the second, that of *stability of position*.

In a structure composed of a series of blocks, or of a series of courses so bonded that each may be considered as one block, which blocks or courses press against each other at plane joints, the two conditions of stability must be fulfilled at each joint.

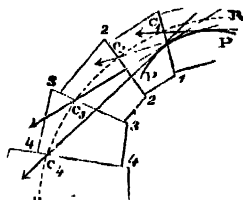


Fig. 123.

Let fig. 123 represent part of such a structure, 1, 1, 2, 2, 3, 3, 4, 4, being some of its plane joints.

Suppose the centre of pressure C_1 of the joint 1, 1, to be known, and also the amount and direction of the pressure, as indicated by the arrow traversing C_1. With that pressure combine the weight of the block 1, 2, 2, 1, together with any other external force which may act on that block; the resultant will be the total pressure to be resisted at the joint 2, 2, which will be given in magnitude, direction, and position, and will intersect that joint in the centre of pressure C_2. By continuing this process there are found the centres of pressure C_3, C_4, &c., of any number of successive joints, and the directions and magnitudes of the resultant pressures acting at those joints.

The magnitude and position of the resultant pressure at any joint whatsoever, and consequently the centre of pressure at that joint, may also be found simply by taking the resultant of all the forces which act on one of the parts into which that joint divides the structure.

The centres of pressure at the joints are sometimes called *centres of resistance*. A line traversing all those centres of resistance, such as the dotted line R R, in fig. 122, has received from Mr. Moseley the name of the "*line of resistance;*" and that author has also shown how in many cases the equation which expresses the form of that line may be determined, and applied to the solution of useful problems.

The straight lines representing the resultant pressures may be all parallel, or may all lie in the same straight line, or may all intersect in one point. The more common case, however, is that in which those straight lines intersect each other in a series of points, so as to form a polygon. A curve, such as P P, in fig. 95, touching all the sides of that polygon, is called by Mr. Moseley the "*line of pressures.*"

The properties which the line of resistance and line of pressures must have, in order that the conditions of stability may be fulfilled, are the following:—

To insure stability of position, *the line of resistance must not deviate from the centre of figure of any joint by more than a certain fraction of the diameter of the joint, measured in the direction of deviation.*

To insure stability of friction, *the normal to each joint must not make an angle greater than the angle of repose with a tangent to the line of pressures drawn through the centre of resistance of that joint.*

The moment of stability of a body or structure supported at a given plane joint is the moment of the couple of forces which must be applied in a given vertical plane to that body or structure in addition to its own weight, in order to transfer the centre of resistance of the joint to the limiting position consistent with stability, and is equal to the product of the weight of the body or structure by the horizontal distance of a vertical line traversing its centre of gravity from the limiting position of the centre of resistance of the base or supporting joint. The applied couple usually consists of the thrust of a frame, or an arch, or the pressure of a fluid, or of a mass of earth, against the structure, together with the equal, opposite, and parallel, but not directly opposed, resistance of the joint to that lateral force.

141. **Transformation of Blockwork Structures.** (*A. M.*, 208, 209.) —If a structure composed of blocks have stability of position when acted on by forces represented by a given system of lines, then will a structure whose figure is a parallel projection of the original structure have stability of position when acted on by forces represented by the corresponding parallel projection of the original system of lines; also, the centres of pressure in the new structure will be the corresponding projections of the centres of pressure in the original structure.

The question, whether the new structure obtained by transformation will possess *stability of friction* is an independent problem.

The application of the principles of the stability of blockwork structures will be illustrated under the head of Masonry.

Section V.—*Of the Strength of Materials in General.*

142.—**Strain, Stress, Strength, Working Load.** (*A. M.*, 244.)—The present section contains a summary of the principles of the strength of materials so far as they relate to questions which arise in designing structures. The rules are given without demonstration, in as small compass as possible, in order to save the necessity of referring, in ordinary cases, to more bulky treatises; and are almost all abstracted and abridged from the treatise already referred to on *Applied Mechanics*, Part II., Chapter III.

The *load*, or combination of external forces, which is applied to any piece in a structure produces *strain*, or alteration of the volumes and figures of the whole piece, and of each of its particles, which is accompanied by *stress* amongst the particles of the piece, being the combination of forces which they exert in resisting the tendency of the load to disfigure and break the piece. If the load is continually increased, it at length produces either *fracture*, or (if the material is very tough and ductile) such a disfigurement of the piece as is practically equivalent to fracture, by rendering it useless.

The **Ultimate Strength or Breaking Load** of a body is the load required to produce fracture in some specified way. The **Proof Strength or Proof Load** is the load required to produce the greatest strain of a specific kind consistent with safety; that is, with the retention of the strength of the material unimpaired. A load exceeding the proof strength of the body, although it may not produce instant fracture, produces fracture eventually by long-continued application and frequent repetition.*

The **Working Load** on each piece of a structure is made less than the proof strength in a certain ratio determined by practical experience, in order to provide for unforeseen contingencies.

* A series of experiments on the effect of the frequent application and removal of a load, were made by the Commissioners on the Application of Iron to Railway Structures, the general results of which were as follows:—

When cast iron bars were exposed to successive transverse blows, each blow producing *one-third* of the ultimate deflection (or deflection immediately before breaking), they bore 4,000 such blows without having their strength impaired; but when the force of each blow produced *one-half* of the ultimate deflection, every bar broke before receiving the 4,000th blow.

When cast iron bars were exposed to successive deflections by means of a cam, of *one-third* of the ultimate deflection, they bore 100,000 such deflections without having their strength impaired; but when each deflection was *one-half* of the ultimate deflection, the bars broke with fewer than 900 deflections.

"In wrought iron bars, no perceptible effect was produced by 10,000 successive deflections, by means of a revolving cam, each deflection being due to half the weight, which, when applied statically, produced a large permanent deflection."

A new series of experiments on the effect of vibratory action and long-continued

Each solid has as many different kinds of strength as there are different ways in which it can be strained or broken, as shown in the following classification :—

Application of Load.	Strain.	Fracture.
Longitudinal......	Extension............	Tearing.
	Compression............	Crushing.
	Distortion	Shearing.
Transverse.	Twisting................	Wrenching.
	Bending................	Breaking across.

143. A **Factor of Safety** (*A.M.*, 247) when not otherwise specified, means the ratio in which the breaking load exceeds the working load.

In fixing factors of safety, a distinction is to be drawn between a *dead load*—that is, a load which is put on by imperceptible degrees, and which remains steady, such as the weight of the structure itself, and a *live load*—that is, a load which is put on suddenly or accompanied with vibration, such as a swift train travelling over a railway bridge, or a force exerted in a moving machine.

Comparing together the results of experiments on the strength of iron, steel, and other materials, and the rules followed in the practice of engineers, the following table gives a fair summary of our knowledge respecting factors of safety :—

changes of load on wrought iron girders, was carried out many years ago by Fairbairn. Those experiments (so far as they had then been carried) were communicated to the British Association at Oxford, in June, 1860 ; and the following is a summary of the results :—

The beam experimented on was a rivetted wrought-iron plate girder.

When the load applied was about *one-fourth* of the breaking weight, the beam withstood 596,790 successive applications of it without visible alteration.

The load was then increased to *two-sevenths* of the breaking weight, and applied 403,210 times, when the beam showed a slight increase of permanent set.

The load was further increased to *two-fifths* of the breaking weight, when the beam broke with the 5,175th application.

The successive applications of the load were accompanied with considerable strains from vibration and impact.

	Dead Load.	Live Load.
For perfect materials and workmanship,	2	4
For good ordinary materials and workmanship:—		
in Metals,............................	3	6
in Timber,............................	4 to 5	8 to 10
in Masonry,............................	4	8

Since the experiments of Fairbairn, Hodgkinson, Rodman, and others, the subject of the behaviour of materials under the action of applied forces has had increasing attention, and now laboratories for the investigation of such subjects are being considered a necessary adjunct to the various schools and universities where Applied Mechanics forms part of the course of study. The phenomena of the Flow of Metals have been specially investigated by Tresca, commencing with his contribution in 1864 entitled *Memoire sur l'Écoulement des Corps Solides*,* and more recently Wohler, Spangenberg, and others, have very fully investigated the effects of repeated applications of given loads to the same test piece. From these experiments it has been determined that the repeated action of forces, the greatest of which is less than the assumed ultimate strength of the piece experimented upon, will cause rupture. This result has been termed the *Fatigue of Metals*. Considering some of the results of such experiments, we find that, unless the applied force is not very far below the ultimate strength of the piece tested, the number of repetitions of the force required to break the piece is very large. Where the test force acted at about three-fourths of the ultimate strength, the test-piece remained unbroken, even after some millions of applications of the test load.

In Britain the Board of Trade has fixed $6\frac{1}{2}$ tons, or 14,560 lbs. per square inch, as the safe working strength for steel of from 28 to 31 tons of ultimate strength per square inch. The American practice seems much the same, as about 15,000 lbs. is their working strength for steel of similar ultimate strength. Where long posts or struts enter into the design of the girder, special formulæ for these compression members have to be used (see Appendix, pp. 795 and 800). Since this weakening effect of *repeated* application of forces to the test-pieces has been more completely understood, the introduction of a factor in formulæ for strength of materials, containing the ratio of the minimum to the maximum force applied, has come into use (see Appendix to Article 378, p. 800). The effect of this is to reduce to some extent the value of the ultimate strength as determined by one application of the load, this result

* *Comptes Rendus*, lix., 1864, and *Journal Franklin Inst.*, li., 1866.

being divided by a suitable factor of safety gives the working strength.

One-fourth or one-fifth of the ultimate strength is usually taken as a safe value for the working strength of wrought iron or steel work.

Where materials are of a crystalline structure, such as cast iron, the liability to fracture by shocks is greater. Thus, if a piece of cast iron be loaded to about half its weight, and at the same time be subjected to a few taps with a hammer, fracture may ensue without any addition to the load: hence in the case of machinery large factors of safety are employed. In designing structures exposed to the severe action of the wind in storms, it is customary in some cases to allow for one-half more than what is due to the dead and live loads, so that there shall be plenty of area to meet the extra or wind stress.

From a Report, drawn up by a Committee composed of engineers and scientific men, as to wind-pressure, the following conclusions have been arrived at:*—

(1.) For railway bridges and viaducts, a maximum wind-pressure of 56 lbs. per square foot should be assumed for the purposes of calculation. (2.) That for girders with closed sides, and as high, or higher, than the top of railway vehicles, the full pressure of 56 lbs. should be employed for the whole vertical surface of one girder, and that where the girder is not as high as the vehicles, the surface should be taken as that of the length of the girder by the height from the bottom thereof to the top of the vehicle. (3.) That for lattice or open girders, the pressure of 56 lbs. should be applied to one girder, as though the girders had closed sides from the level of the rails to the top of the train, and the same pressure to be applied to the actual ironwork area below the level of the rails and above the top of the train. The pressures to be applied to the inner or leeward girder, one only, in addition to the above, relate only to the actual vertical area of ironwork below the rails and above the train, and are: (*a.*) 28 lbs. per square foot, when the open spaces are not more than two-thirds of the area included within the outline of the girder; (*b.*) 42 lbs., when the open spaces are between two-thirds and three-fourths the whole outline area; and (*c.*) 56 lbs., when the open spaces are greater than three-fourths the whole area. (4.) The pressure on arches and piers should be ascertained in conformity with these rules. And (5.) a factor of 4 should be employed in all these cases in calculating the necessary strength, except when wind pressure is counteracted by gravity only, then a factor of 2 is considered sufficient.

* See *The Engineer* of 26th August, 1881.

When the working load is partly dead and partly live, multiply each part of the load by its proper factor of safety, and add together the products; the sum will be the ultimate or breaking load to which the piece or structure is to be adapted.

144. The **Proof** or **Testing** by experiment of the strength of a piece of material is to be conducted in two different ways, according to the object in view.

I. If the piece is to be *afterwards used*, the testing load must be so limited that there shall be no possibility of its impairing the strength of the piece; that is, it must not exceed the proof strength, being from one-third to one-half of the ultimate strength. About double of the working load is in general sufficient. Care should be taken to avoid vibrations and shocks when the testing load approaches near to the proof strength.

II. If the piece is to be *sacrificed* for the sake of ascertaining the strength of the material, the load is to be increased by degrees until the piece breaks, care being taken, especially when the breaking point is approached, to increase the load by small quantities at a time, so as to get a sufficiently precise result.

The *proof strength* requires much more time and trouble for its determination than the ultimate strength. One mode of approximating to the proof strength of a piece is to apply a moderate load and remove it, apply the same load again and remove it, two or three times in succession, observing at each time of application of the load, the *strain* or alteration of figure of the piece when loaded, by stretching, compression, bending, distortion, or twisting, as the case may be. If that alteration does *not sensibly increase* by repeated applications of the same load, the load is within the limit of proof strength. The effects of a greater and a greater load being successively tested in the same way, a load will at length be reached whose successive applications produce increasing disfigurements of the piece; and this load will be greater than the proof strength, which will lie between the last load and the last load but one in the series of experiments.

It was formerly supposed that the production of a *set*, that is, a disfigurement which continues after the removal of the load, was a test of the proof strength being exceeded; but Mr. Hodgkinson showed that supposition to be erroneous, by proving that in most materials a set is produced by almost any load, how small soever.

The strength of bars and beams to resist breaking across, and of axles to resist twisting, can be tested by the application of known weights either directly or through a lever.

To test the tenacity of rods, chains, and ropes, and the resistance of pillars to crushing, more powerful and complex mechanism is required. The apparatus most commonly employed is the

hydraulic press. In computing the stress which it produces, no reliance ought to be placed on the load on the safety valve, or on a weight hung to the pump handle, as indicating the intensity of the pressure, which should be ascertained by means of Bourdon's gauge. This remark applies also to the proving of boilers by water pressure.

From experiments made by Messrs. More of Glasgow, and by the Author, it appears, that in experiments on the tension and compression of bars, about *one-tenth* should be *deducted* from the pressure in the hydraulic press for the friction of the press plunger.

The measurement of tension and compression by means of the hydraulic press is but a rough approximation at the best. It may be sufficient for an immediate practical purpose; but for the exact determination of general laws, although the load may be applied at one end of the piece to be tested by means of a hydraulic press, it ought to be resisted and measured at the other end by means of a combination of levers, such as that used at Woolwich Dockyard, and described by Mr. Barlow in his work *On the Strength of Materials*.

145. **Co-efficients or Moduli of Strength** are quantities expressing the *intensity* of the stress under which a piece of a given material gives way when strained in a given manner; such intensity being expressed in units of weight for each unit of sectional area of the layer of particles at which the body first begins to yield. In Britain, the ordinary unit of intensity employed in expressing the strength of materials is the *pound avoirdupois on the square inch*. As to other units, see Article 106, p. 161.

Co-efficients of strength are of as many different kinds as there are different ways of breaking a body. Their use will be explained in the sequel. Tables of their values are given at the end of the volume.

Co-efficients of strength, when of the same kind, may still vary according to the direction in which the stress is applied to the body. Thus the tenacity, or resistance to tearing, of most kinds of wood, is much greater against tension exerted along than across the grain.

146. **Stiffness or Rigidity, Pliability, their Moduli or Co-efficients.**—Rigidity or stiffness is the property which a solid body possesses, of resisting forces tending to change its figure. It may be expressed as a quantity, called a *modulus or co-efficient of stiffness*, by taking the ratio of the intensity of a given stress of a given kind, to the strain, or alteration of figure, with which that stress is accompanied:—that strain being expressed as a quantity by dividing the alteration of some dimension of the body by the original length of

that dimension. In most substances which are used in construction, the moduli of stiffness, though not exactly constant, are nearly constant for stresses not exceeding the proof strength.

The reciprocal of a modulus of stiffness may be called a "*modulus of pliability;*" that is to say,

$$\text{Modulus of Stiffness} = \frac{\text{Intensity of Stress}}{\text{Strain}};$$

$$\text{Modulus of Pliability} = \frac{\text{Strain}}{\text{Intensity of Stress}}.$$

The use of specific moduli of stiffness will be explained in the sequel. Tables of their values are given at the end of the volume.

147. The **Elasticity of a Solid** (*A. M.*, 236 to 238, 243, 248 to 263) consists of stiffness, or resistance to change of figure, combined with the power of recovering the original figure when the straining force is withdrawn. If that recovery is perfect and exact, the body is said to be "*perfectly elastic;*" if there is a "*set,*" or permanent change of figure, after the removal of the straining force, the body is said to be "imperfectly elastic." The elasticity of no solid substance is absolutely perfect, but that of many substances is nearly perfect when the stress does not exceed the proof strength, and may be made sensibly perfect by restricting the stress within small enough limits.

Moduli or *Co-efficients of Elasticity* are the values of moduli of stiffness when the stress is so limited that the value of each of those moduli is sensibly constant, and the elasticity of the body sensibly perfect. It can be shown that in a homogeneous solid, there may be *twenty-one* independent co-efficients of elasticity,* which, in a solid that is equally elastic in all directions, are reduced to *two*—viz., the co-efficient of direct elasticity, or resistance to direct lengthening and shortening, and the co-efficient of resistance to distortion.

The *General Problem of the Internal Equilibrium of an Elastic Solid* is this:—Given the free form of a solid, the values of its co-efficients of elasticity, the attractions acting on its particles, and the stresses applied to its surface; to find its change of form, and the strains of all its particles.† This problem is to be solved, in general, by the aid of an ideal division of the solid into molecules, rectangular in their free state, and referred to rectangular co-ordinates. Some particular cases are most readily solved by means

* See *Phil. Trans.*, 1856-7.

† See Lamé, *Leçons sur la Théorie Mathématique de l'Elasticité des Corps.*

of spherical, cylindrical, or otherwise curved co-ordinates. The general problem is of extreme complexity, and its complete solution has not yet been obtained in a practically available form; but the cases which occur in practice, and to which the remainder of this section relates, can be solved with sufficient accuracy by comparatively simple approximate methods. Most of those approximate methods are analogous to the "method of sections" described in its application to framework in Article 122, p. 184. The body under consideration is conceived to be divided into two parts by an ideal plane of section; the external forces and couples acting on one of those two parts are computed; and they must be equal and opposite to the forces and couples resulting from the *entire* stress at the ideal sectional plane, which is so found. Then as to the *distribution* of that stress, direct and shearing, some law is assumed, which, if not exactly true, is known either by experiment or by theory, or by both combined, to be a sufficiently close approximation to the truth.

Except in a few comparatively simple cases, the strict method of investigation, by means of the equations of internal equilibrium, has hitherto been used only as a means of determining whether the ordinary approximative methods are sufficiently close.

148. **Resilience, or Spring** (*A. M.*, 244), is the quantity of *mechanical work* required to produce the proof-stress on a given piece of material, and is equal to the product of the *proof strain*, or alteration of figure, into the mean load which acts during the production of that strain; that is to say, in general, very nearly one-half of the proof load.

149. **Resistance of Bars to Stretching and Tearing.** (*A. M.*, 265 to 269.)—The ultimate strength or breaking load of a bar exposed to direct and uniform tension is the product of the area of cross-section of the bar into the *tenacity* of the material. Therefore let

P denote the breaking load, in pounds;

S the area of section, in square inches;

f the tenacity, in pounds on the square inch; then

$$P = f S ; \; S = \frac{P}{f} \qquad \text{(1.)}$$

The *elongation* of the bar under any load P′ not exceeding the proof load is found as follows:—

Let x denote the original length of the bar, Δx the elongation, and

Δx

the *proportion* which that elongation bears to the original length of the bar, being the numerical measure of the strain.

Let $p = P \div S$ denote the intensity of the stress, and E, the *modulus of direct elasticity*, or resistance to stretching. Then

$$\alpha = \frac{p}{E} \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (2.)$$

Let f' denote the *proof tension* of the material, so that f' S is the proof load of the bar; then the *proof strain*, or proportionate elongation under the proof load, is $f' \div E$.

The **Resilience** or **Spring** of the bar, or the work performed in stretching it to the limit of proof strain, is computed as follows:—x being the length, as before, the elongation of the bar under the proof load is $f' x \div E$. The force which acts through this space has for its least value 0, for its greatest value $P = f' S$, and for its mean value $f' S \div 2$; so that the work performed in stretching the bar to the proof strain is

$$\frac{f' S}{2} \cdot \frac{f' x}{E} = \frac{f'^2}{E} \cdot \frac{S x}{2} \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (3.)$$

The co-efficient $f'^2 \div E$, by which one-half of the volume of the bar is multiplied in the above formula, is called the MODULUS OF RESILIENCE.

A *sudden pull* of $f' S \div 2$, or *one-half of the proof load*, being applied to the bar, will produce the *entire proof strain* of $f' \div E$, which is produced by the *gradual* application of the proof load itself; for the work performed by the action of the constant force $f' S \div 2$, through a given space, is the same with the work performed by the action, through the same space, of a force increasing at an uniform rate from 0 up to f' S. Hence a bar, to resist with safety the sudden application of a given pull, requires to have twice the strength that is necessary to resist the gradual application and steady action of the same pull.

Tables of the tenacity and of the modulus of direct elasticity of various substances are given at the end of the volume.

150. **Cylindrical Boilers and Pipes.** (*A. M.*, 271.)—Let r denote the radius of a thin hollow cylinder, such as the shell of a high pressure boiler;

t the thickness of the shell;

f the tenacity of the material, in pounds per square inch;

p the intensity of the pressure, in pounds per square inch, required to burst the shell. This ought to be taken at SIX TIMES the effective working pressure—*effective pressure* meaning the excess of the pressure from within above the pressure from without, which

last is usually the atmospheric pressure, of 14·7 lbs. on the square inch or thereabouts.

Then

$$p = \frac{f\,t}{r}; \quad \text{.........................(1.)}$$

and the proper proportion of thickness to radius is given by the formula,—

$$\frac{t}{r} = \frac{p}{f}. \quad \text{.............................(2.)}$$

(See p. 799.)

151. **Spherical Shells,** such as the ends of "egg-ended" cylindrical boilers, the tops of steam domes, &c., are *twice as strong* as cylindrical shells of the same radius and thickness.

Suppose a shell of the figure of a segment of a sphere to have a circular flange round its base, through which it is bolted to a flange upon a cylindrical shell, or upon another spherical shell.

Let r denote the radius of the sphere, in inches;

r', the radius of the circular base of the segmental shell, in inches;

p, the bursting pressure, in lbs. on the square inch;

then the number and dimensions of the bolts by which the flange is held should be such, that the load required to tear them asunder all at once shall be

$$3 \cdot 1416\, r'^2 p; \quad \text{........................(1.)}$$

and the flange itself should require, in order to crush it, the following thrust in the direction of a tangent to it:—

$$\frac{1}{2} p\, r' \cdot \sqrt{r^2 - r'^2}. \quad \text{......................(2.)}$$

If the segment is a complete hemisphere, $r' = r$, and the last expression becomes $= 0$.

152. **Thick Hollow Cylinder.** (*A. M.*, 273.)—The assumption that the tension in a hollow cylinder is uniformly distributed throughout the thickness of the shell is approximately true only when the thickness is small as compared with the radius.

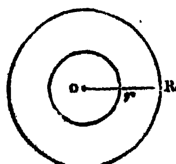


Fig. 124.

Let R represent the external and r the internal radius of a thick hollow cylinder, such as a hydraulic press, the tenacity of whose material is f, and whose bursting pressure is p. Then we must have

$$\frac{R^2 - r^2}{R^2 + r^2} = \frac{p}{f}, \quad \text{(1.)}$$

and, consequently,

$$\frac{R}{r} = \sqrt{\left(\frac{f+p}{f-p}\right)}; \quad \text{(2.)}$$

by means of which formula, when r, f, and p, are given, R may be computed.

153. **Thick Hollow Sphere.** (*A. M.*, 275.)—In this case, using the same symbols as in the last Article, the following formulæ give the ratios of the bursting pressure to the tenacity, and of the external to the internal radius:—

$$\frac{p}{f} = \frac{2R^3 - 2r^3}{R^3 + 2r^3}; \quad \text{(1.)}$$

$$\frac{R}{r} = \sqrt[3]{\left(\frac{2f + 2p}{2f - p}\right)}. \quad \text{(2.)}$$

154. **Lateral Pliability, Cubic Compressibility.**—When the side surfaces of a bar are free, the application of tension to its ends causes it to contract in thickness as well as to extend in length; and the application of pressure to its ends causes it to expand in thickness as well as to contract in length. This property, which may be called "Lateral Pliability," exists to the greatest possible extent in perfect fluids, whose parts yield laterally to the slightest longitudinal stress, and is least in those solids which are least capable of changes of figure.

If a solid bar has the alteration of its transverse dimensions prevented or resisted by any means, it yields less longitudinally to a longitudinal stress than it does when it is free to yield laterally; in other words, its direct or longitudinal stiffness may be increased; and that according to laws whose mathematical expression will presently be given. Its strength is increased also; but in what proportion is not yet known precisely.

Let p denote the intensity of a longitudinal stress, not exceeding the limit below which moduli of stiffness are constant; and when the bar is free to alter its lateral dimensions, let,

$\alpha = \frac{p}{E}$ denote the fraction of its original length by which its length is altered, and

$\beta = \frac{p}{D}$, the fraction of its original diameter by which its diameter is altered in the contrary direction to its length; then we have

$$p = A\alpha - B\beta; \quad \text{(1.)}$$

A and B being two co-efficients whose relations to E and D are expressed by the two following pairs of equations:—

$$\left.\begin{aligned} A &= \frac{E D (D - E)}{D^2 - E D - 2 E^2} = E + \frac{2 E^3}{D^2 - E D - 2 E^2}, \\ B &= \frac{E^2 D}{D^2 - E D - 2 E^2}; \ \frac{B}{A} = \frac{E}{D - E}; \end{aligned}\right\} (2.)$$

$$E = A - \frac{2 B^2}{A + B}; \ D = A - 2 B + \frac{A^2}{B}; \ \frac{D}{E} = \frac{A + B}{B}. \ (3.)$$

The co-efficient A represents the resistance of the bar to direct elongation or compression when $\beta = 0$; that is, when the transverse dimensions of the bar are prevented from changing.*

The resistance to alteration of volume bears the following relations to the before-mentioned co-efficients. Let p denote the uniform intensity of a pressure or tension applied *all over the surface* of a body, and δ the fraction of the original volume by which the volume is diminished or increased; then the *cubic elasticity* is

$$\frac{p}{\delta} = \frac{A + 2 B}{3} = \frac{3 D - 6 E}{D E}; \ldots\ldots\ldots\ldots\ldots (4.)$$

and the reciprocal of this is the *cubic compressibility.*

The values of A and B have been ascertained for a few substances only. For brass and crystal, according to M. Wertheim's experiments (*Annales de Chimie*, third series, vol. xxiii.), the following ratios hold very nearly:—

$$A \div B = 2; \ D \div E = 3; \ A \div E = \frac{3}{2}; \ B \div E = \frac{3}{4}; \ldots (5.)$$

and consequently, for those substances,

$$\frac{p}{\delta} = \frac{2}{3} A = E. \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (6.)$$

155. **Heights of Moduli of Stiffness and Strength.**—The term "*Height*," as applied to a given modulus, whether of stiffness or

* A is the co-efficient of elasticity deduced from experiments on the velocity of sound in a solid body of large transverse dimensions, by means of the following formula, in which v is the velocity of sound in feet per second, and w the weight of a cubic foot of the body;

$$A \text{ in lbs. on the square foot} = \frac{w v^2}{32 \cdot 2}.$$

When the transverse dimensions of the body are narrow, this formula gives results lying between the value of A and that of E.

of strength, means the height of an imaginary column of the substance to which the modulus belongs, whose weight would cause a pressure on its base, equal in intensity to the stress expressed by the given modulus. Hence,

Height of a modulus in feet

$$= \frac{\text{Modulus in lbs. on the square foot}}{\text{Heaviness of substance in lbs. to the cubic foot}}$$

$$= \frac{\text{Modulus in lbs. on the square inch}}{\text{Weight of 12 cubic inches of substance}};$$

Height of a modulus in inches

$$= \frac{\text{Modulus in lbs. on the square inch}}{\text{Weight of a cubic inch of substance}}.$$

156. **Resistance to Shearing and Distortion.** (*A. M.*,' 278 to 281.)—In structures, many cases occur in which the principal pieces, such as plates, links, bars, or beams, being themselves subjected to a direct pull, are connected with each other at their joints by fastenings, such as rivets, bolts, pins, keys, or screws, which are under the action of a shearing force, tending to make them give way by the sliding of one part over another.

The present Article refers to those cases only in which the shearing stress on a body is uniform in direction and in intensity. The effects of shearing stress varying in intensity will be considered under the head of Resistance to Bending, which is in general accompanied by such a stress; and the effects of shearing stress varying in direction as well as in intensity under the head of Resistance to Torsion.

To insure uniform distribution of the stress, it is necessary that the rivet or other fastening should fit so tight in its hole or socket, that the friction at its surface may be at least of equal intensity to the shearing stress. When this condition is fulfilled, the intensity of that stress is represented simply by $F \div S$, F being the shearing force, and S the area which resists it.

In consequence of the relation between shearing stress and direct stress, stated in Article 108, Division II., p. 167, it appears that a body may give way to a shearing stress either by actual shearing, at a plane parallel to the direction of the shearing force, or by tearing in a direction making an angle of 45° with that force.

When a shearing stress does not exceed the limit within which moduli of stiffness are sensibly constant, it produces distortion of the body on which it acts. Let q denote the intensity of a shearing stress applied to the four lateral faces of an originally square

prismatic particle, so as to distort it; and let ν be the ***distortion,*** expressed by the *tangent of the difference between each of the distorted angles of the prism and a right angle;* then

$$q = C, \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (1.)$$

is the ***modulus of transverse elasticity***, or ***resistance to distortion.***

One mode of expressing the distortion of an originally square prism is as follows:—Let α denote the proportionate elongation of one of the diagonals of its end, β the proportionate shortening of the other; then the distortion is

$$\nu = \alpha + \beta.$$

The co-efficient C is necessarily related to those mentioned in Article 154, pp. 229, 230, in the following manner:—

$$C = \frac{A - B}{2} = \frac{E\,D}{2\,(D + E)}. \quad \ldots\ldots\ldots\ldots (2.)$$

For brass and crystal, according to M. Wertheim's experiments, we have

$$C = \frac{A}{4} = \frac{B}{2} = \frac{3\,E}{8} = \frac{D}{8}. \quad \ldots\ldots\ldots\ldots (3.)$$

The co-efficient C expresses the quality of rigidity, or resistance to change of figure, which distinguishes solids from fluids.

In a perfect fluid, the following relations hold:—

$$\left.\begin{array}{l} C = 0;\ E = 0;\ D = 0; \\ A = B = \frac{p}{\delta}\ \text{(the cubic elasticity).} \end{array}\right\} \quad \ldots\ldots\ldots\ldots (4.)$$

The general effect of heat on solid bodies is to diminish C and increase $p \div \delta$.

157. **Resistance to Compression and Direct Crushing.** (*A. M.*, 282 to 286.)—Resistance to *Longitudinal Compression*, when the proof stress is not exceeded, is sensibly equal to the resistance to stretching, and is expressed by the same modulus. When that limit is exceeded, it becomes irregular.

Crushing, or breaking by compression, is not a simple phenomenon like tearing, but is more or less complex and varied, according to the nature of the substance.

The present Article has reference to direct crushing only, and is limited to those cases in which the pillars, blocks, struts, or rods,

along which the pressure acts are not so long in proportion to their diameter as to have a sensible tendency to give way by bending sideways. Those cases comprehend—

Stone and brick pillars and blocks of ordinary proportions;

Pillars, rods, and struts of cast iron, in which the length is not more than five times the diameter, approximately;

Pillars, rods, and struts of wrought iron, in which the length is not more than ten times the diameter, approximately;

Pillars, rods, and struts of dry timber, in which the length is not more than about twenty times the diameter.

Let P denote the *crushing load* of the piece;

S the area of its transverse section in square inches;

f the resistance of the material to crushing, in lbs. on the square inch; then *if the load is uniformly distributed*,

$$P = f S \quad \ldots\ldots\ldots\ldots \quad (1.)$$

A table of the resistance of materials to direct crushing, in lbs. on the square inch, is given at the end of the volume.

If the load is not uniformly distributed over the transverse section of the pillar, the strength of the pillar is diminished in the same ratio in which the mean intensity of the stress is less than the maximum intensity. To find that ratio, it is sufficiently near the truth for practical purposes to consider the stress as "*uniformly varying*." (See Article 106, Division II., p 163, equations 5, 6, 7.) Suppose the pillar to be cylindrical, square, or of a regular polygonal figure in cross-section. Let x_0 be the greatest deviation of the centre of pressure from the centre of figure in any cross-section; that is, the greatest deviation of the line of action of the load from the axis of the pillar.

Let x_1 be the distance of the point of greatest stress from the axis of the pillar, that is, the semidiameter of the pillar in the direction in which the load deviates from the axis.

Let $I = \int x^2 \, y \, d x$ denote what is called the "moment of inertia" of the cross-section of the pillar.

Then the crushing load is,

$$P = \frac{f S}{1 + \frac{x_0 x_1 S}{I}} \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots \quad (2.)$$

The following are some of the values of $\frac{x_1 S}{I}$ in the preceding formula: the "*neutral axis*" meaning the diameter to which the deviation x_0 is perpendicular.

Figure of Cross-Section.	$\frac{x_1 S}{\mathrm{I}}$.
I. Rectangle, $h\,b$; b, neutral axis, II. Square, h^2, ...	.... $\frac{6}{h}$
III. Ellipse: neutral axis, b; other axis, h; IV. Circle: diameter, h, ...	$\frac{8}{h}$
V. Hollow rectangle: outside dimensions, h, b; inside dimensions, h', b'; neutral axis, b,	$\frac{6\,h\,(h\,b - h'\,b')}{h^3\,b - h'^3\,b'}$.
VI. Hollow square, $h^2 - h'^2$, ...	$\frac{6\,h}{h^2 + h'^2}$
VII. Circular ring: diameter, outside, h; inside, h',	$\frac{8\,h}{h^2 + h'^2}$

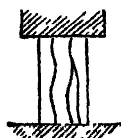
It is often advisable, especially in masonry, so to limit the deviation of the centre of pressure from the axis of the pillar, that there shall be *no tension* on any part of it. This condition is fulfilled when the least pressure is positive, or nothing, and the greatest stress not more than double of the mean stress, so that $P \angle f S \div 2$; and consequently, when

$$x_0 \angle \frac{\mathrm{I}}{x_1 S} \dots\dots\dots\dots\dots\dots \quad (3.)$$

the reciprocal of the quantity of whose values examples have just been given.

The modulus of resistance to direct crushing, as the tables show, often differs considerably from the tenacity. The nature and amount of those differences depend mainly on the modes in which the crushing takes place. These may be classed as follows:—

I. *Crushing by splitting* (fig. 121), into a number of nearly prismatic fragments, separated by smooth surfaces whose general direction is nearly parallel to the direction of the load, is characteristic of hard homogeneous substances of a glassy texture, such as vitrified bricks.


Fig. 125.

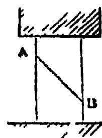


Fig. 126.

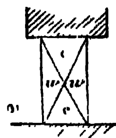
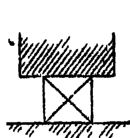


Fig. 127.


Fig. 128.

II. *Crushing by shearing or sliding* of portions of the block along oblique surfaces of separation is characteristic of substances of a

granular texture, like cast iron, and most kinds of stone and brick. Sometimes the sliding takes place at a single plane surface, like A B in fig. 126; sometimes two cones or pyramids are formed, like c, c, in fig. 127, which are forced towards each other, and split or drive outwards a number of wedges surrounding them, like *w*, *w*, in the same figure. Sometimes the block splits into four wedges, as in fig. 128.

The surfaces of shearing make an angle with the direction of the crushing force, which Mr. Hodgkinson (who first fully investigated those phenomena) found to have values depending on the kind and quality of material. For different qualities of cast iron, for example, that angle ranges from 42° to 32°. The greatest intensity of shearing stress is on a plane making an angle of 45° with the direction of the crushing force; and the deviation of the plane of shearing from that angle shows that the resistance to shearing is not purely a cohesive force, independent of the normal pressure at the plane of shearing, but consists partly of a force analogous to friction, increasing with the intensity of the normal pressure.

Mr. Hodgkinson considers that in order to determine the true resistance of substances to direct crushing, experiments should be made on blocks in which the proportion of length to diameter is not less than that of 3 to 2, in order that the material may be free to divide itself by shearing. When a block which is shorter in proportion to its diameter is crushed, the friction of the flat surfaces between which it is crushed has a perceptible effect in *holding its parts together*, so as to resist their separation by shearing; and thus the apparent strength of the substance is increased beyond its real strength.

In all substances which are crushed by splitting and by shearing, the resistance to crushing considerably exceeds the tenacity, as the tables show. The resistance of cast iron to crushing, for example, was found by Mr Hodgkinson to be somewhat more than *six* times its tenacity.

III. *Crushing by bulging*, or lateral swelling and spreading of the block which is crushed, is characteristic of ductile and tough materials, such as wrought iron. Owing to the gradual manner in which materials of this nature give way to a crushing force, it is difficult to determine their resistance to that force exactly. That resistance is in general less, and sometimes considerably less, than the tenacity. In wrought iron, the resistance to the direct crushing of short blocks, as nearly as it can be ascertained, is from $\frac{2}{3}$ to $\frac{4}{5}$ of the tenacity.

IV. *Crushing by buckling or crippling* is characteristic of fibrous substances, under the action of a thrust along the fibres. It consists

in a lateral bending and wrinkling of the fibres, sometimes accompanied by a splitting of them asunder. It takes place in timber, in plates of wrought iron, and in bars longer than those which give way by bulging. The resistance of fibrous substances to crushing is in general considerably less than their tenacity, especially where the lateral adhesion of the fibres to each other is weak compared with their tenacity. The resistance of most kinds of timber to crushing, when dry, is from $\frac{1}{2}$ to $\frac{2}{3}$ of the tenacity. Moisture in the timber weakens the lateral adhesion of the fibres, and reduces the resistance to crushing to about one half of its amount in the dry state.

158. **Crushing by Cross-breaking** is the mode of fracture of columns and struts in which the length greatly exceeds the diameter. Under the breaking load, they yield sideways, and are broken across like beams under a transverse load.

The laws of this mode of breaking were investigated experimentally by Mr. Hodgkinson. The following are his formulæ for cast iron cylindrical pillars:—

When the length is not less than thirty times the diameter; for solid pillars, let h be the diameter *in inches*, L the length *in feet*, and A a constant multiplier; then

$$\text{Breaking load} = A\ h^{3.6} \div L^{1.7}; \dots\dots\dots\dots\dots (1.)$$

for hollow pillars, let h be the external and h' the internal diameter, in inches; then

$$\text{Breaking load} = A\ (h^{3.6} - h'^{3.6}) \div L^{1.7}. \dots\dots\dots (2.)$$

The values of the co-efficient A are as follows:—

	Tons.
for solid pillars with rounded ends,......	14·9
" " " fixed ends,........................	44·16
for hollow pillars with rounded ends,..	13·0
" " " fixed ends, ...	44·3.

The strength of a pillar with one end fixed and the other rounded, is nearly a mean between the strength of two pillars of the same dimensions, one with both ends fixed, and the other with both ends rounded.

When the length is less than thirty times the diameter; let b denote the breaking load of the pillar, computed by the preceding formulæ:—

Let c be the crushing load of a short block of the same sectional area, = 49 tons × sectional area in square inches; then

$$\text{Actual breaking load of pillar} = \frac{4\,b\,c}{4\,b + 3\,c}. \quad \ldots\ldots(3.)$$

The following are formulæ deduced by Mr. Lewis Gordon from Mr. Hodgkinson's experiments:—

Let P be the crushing load of a long rod or pillar, in lbs.;
S the sectional area of material in it, in square inches;
l, its length, }
h, its least external diameter, } both in the same units of measure.
Then, approximately—

$$P = \frac{f\,S}{1 + a \cdot \frac{l^2}{h^2}} \quad \ldots\ldots(4.)$$

The following are the values of f and a, for pillars fixed at both ends, by having flat capitals and bases:—

	f, lbs. per inch.	a.
Wrought iron (rectangular struts);	36,000	$\frac{1}{3,000}$
Cast iron (hollow cylindrical pillars);	80,000	$\frac{1}{800}$
Timber (rectangular pillars);	7,200	$\frac{1}{250}$
Stone and Brick (rectangular pillars);	see tables, pp. 361, 769	$\frac{1}{600}$

For a pillar rounded or jointed at both ends,

$$P = \frac{f\,S}{1 + 4\,a\,\frac{l^2}{h^2}} \quad \ldots\ldots(5.)$$

(See p. 795.)

In wrought iron framework and machinery, the bars which act as struts, in order that they may have sufficient stiffness, are made of various forms in cross-section, well known as "angle iron," "channel iron," "T-iron," "double T-iron," &c. As to the quantity to be put instead of h in such cases, see Article 366, page 522.

Wrought iron cells are rectangular tubes (generally square) usually composed of four plate iron sides, rivetted to angle iron bars at the corners. The *ultimate resistance* of a single square cell to crushing by the buckling or bending of its sides, when the thick-

ness of the plates is ***not less than one-thirtieth of the diameter of the cell***, as determined by Fairbairn and Hodgkinson, is

27,000 lbs. per square inch section of iron;

but when a number of cells exist side by side, their stiffness is increased, and their ultimate resistance to a thrust may be taken at

33,000 to 36,000 lbs. per square inch section of iron.

The latter co-efficients apply also to cylindrical cells.

For further information respecting the application of equations (4) and (5), see pages 520 to 524. It is to be observed that these formulæ, as they stand, are strictly applicable only to cases in which the line of action of the thrust sensibly coincides with the *axis* of the strut; that is, a straight line traversing the centres of its cross-sections. When the line of action deviates from the *axis* of the strut, the following modification is to be made: let x be the greatest deviation, r, the *radius of gyration* of the cross-section of the strut (as to which, see page 523); then to the divisor in each of the formulæ (4) and (5) add the following quantity:—

$$\frac{x\,h}{2\,r^2} \quad \text{...(6.)}$$

The values of this, for some ordinary forms of section, are:—

Solid rectangle,................................$\frac{6\,x}{h}$;

Solid cylinder,..................................$\frac{8\,x}{h}$;

Thin hollow cylinder,........................$\frac{4\,x}{h}$.

(See also pages 233, 234.)

159. **Resistance to Collapsing.**—When a thin hollow cylinder is pressed from without, it gives way by *collapsing*, under a pressure whose intensity has been found by Fairbairn (*Philos. Trans.*, 1858)* to vary nearly according to the following laws:—

Inversely as the length;

Inversely as the diameter;

Directly as a function of the thickness, which is very nearly the power whose index is 2·19; but which for ordinary practical purposes may be treated as sensibly equal to the *square* of the thickness.

The following formula gives approximately the *collapsing pressure* p, in lbs. on the square inch, of plate iron flue with butt-joints,

* See also *Useful Information for Engineers*, second series, 1860.

whose length l, diameter d, and thickness t, are all expressed in *the same units of measure*:—

$$p = 9{,}672{,}000 \frac{t^2}{l\,d} \text{.......................(1.)}$$

Let t and d be expressed in inches, and let L be the length in feet; the above formula becomes

$$p = 806{,}000 \frac{t^2}{L\,d} \text{.........................(2.)}$$

Fairbairn having strengthened tubes by rivetting round them rings of T-iron, or angle iron, at equal distances apart, found that their strength is that corresponding to the length *from ring to ring*.

He also found that the collapsing pressure of a tube of an elliptic form of cross-section is found approximately by substituting for d, in the preceding formulæ, the diameter of the osculating circle at the flattest part of the ellipse; that is, let a be the greater, and b the less *semi-axis* of the ellipse; then we are to make

(See p. 799).

$$d = \frac{2\,a^2}{b} \text{............................(3.)}$$

160. **Action of a Transverse Load on a Beam.** (*A. M.*, 288.)— It has already been shown, in Article 112, p. 174, how to determine the proportions between the resultant of the gross load of a beam and the two forces which support it. In the present Article those cases alone will be considered in which the loading and supporting forces are parallel to each other, and in one plane.

In Article 122, p. 184, it has been shown how to determine the resistances exerted by the pieces of a frame which are cut by an ideal sectional plane, in terms of the forces and couples which act on one of the portions into which that plane of section divides the frame.

The method followed in determining the effect of a transverse load on a continuous beam is similar; except that the resistance at the plane section, which is to be determined, does not consist of a finite number of forces acting along the axes of certain bars, but of a distributed stress, acting with various intensities, and, it may be, in various directions, at different points of the section of the beam.

In what follows, the load of the beam will be conceived to consist of weights acting vertically downwards, and the supporting forces will also be conceived to be vertical. The longitudinal axis of the beam being perpendicular to the applied forces, will accord-

ingly be horizontal. The conclusions arrived at are applicable to cases in which the axis of the beam and the direction of the applied forces are inclined, so long as they are perpendicular to each other.

Let any point in the longitudinal axis be taken as the origin of co-ordinates; and at a given horizontal distance x' from that origin, conceive a vertical section perpendicular to the longitudinal axis to divide the beam into two parts.

Let F denote the resultant of all the vertical forces, whether loading or supporting, which act on the part of the beam to the left of the vertical plane of section, and let x'' be the horizontal distance of the line of action of that resultant from the origin.

If the beam is strong enough to sustain the forces applied to it, there will be a *shearing stress* equal and opposite to F, distributed (in what manner will afterwards appear) over the given vertical section; and that shearing stress, or vertical resistance, will constitute, along with the resultant applied force F, a couple whose moment is

$$M = F(x'' - x') \text{ .... (I.)}$$

This is called the *bending moment* or *moment of flexure* of the beam at the vertical section in question; it is resisted by the direct stress at that section, in a manner to be explained in the sequel; and it tends to make the originally straight longitudinal axis of the beam become concave in the direction towards which the resultant applied force F acts.

The mathematical process for finding F and M at any given cross-section of a beam, though always the same in principle, may be varied considerably in detail. The following is on the whole the most convenient way of conducting it.

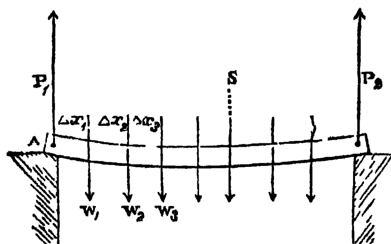


Fig 129.

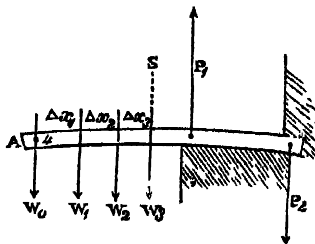


Fig. 130.

Fig. 129 represents a beam *supported* at both ends, and loaded between them. Fig. 130 represents a beam *supported* and *fixed* at one end, and loaded on a projecting portion. P_1, P_2, represent in

each case the supporting forces; in fig. 129, W_1, W_2, W_3, &c., represent portions of the load; in fig. 130, W_0 represents the endmost portion of the load, and W_1, W_2, W_3, other portions; in both figures, Δx_1, Δx_2, Δx_3, &c., denote the lengths of the parts into which the lines of action of the portions of the load divide the horizontal axis of the beam.

The figures represent the load as applied at detached points; but when it is continuously distributed, the length of any indefinitely short portion of the beam will be denoted by $d\,x$, the intensity of the load upon it *per unit of length* by w, and the amount of the load upon it by $w\,d\,x$.

The process to be gone through will then consist of the following steps:—

STEP I. *To find the Supporting Forces.*—Assume any convenient point in the horizontal axis as origin of co-ordinates, and find the distance x_0 of the resultant of the load from it, by the method of Article 97, Division VIII., p. 143, for forces acting through detached points, and by the method of Article 104, Division V., p. 153, for a continuously distributed load; that is to say,

$$\left.\begin{array}{l} x_0 = \dfrac{\Sigma \cdot x\,W}{\Sigma \cdot W}; \text{ or} \\[2ex] x_0 = \dfrac{\int x\,w\,d\,x}{\int w\,d\,x} \end{array}\right\} \qquad (2.)$$

Then, as in Article 112, p. 174, find the two supporting forces, P_1 and P_2; that is to say, if x_1 and x_2 be the distances of the points of support from the origin, with their proper signs, make

$$\left.\begin{array}{l} P_1 = \dfrac{x_2 - x_0}{x_2 - x_1} \Sigma \cdot W \ (\text{or} \int w\,d\,x); \\[2ex] P_2 = \dfrac{x_0 - x_1}{x_2 - x_1} \Sigma \cdot W \ (\text{or} \int w\,d\,x). \end{array}\right\} \qquad (3.)$$

STEP II. *To find the Shearing Forces at a Series of Sections*—In what position soever the origin of co-ordinates may have been during the previous step, assume it now, in a beam supported at both ends, to be at one of the points of support (as A, fig. 129), and in a beam fixed at one end, to be at the loaded point farthest from the fixed end (as A, fig. 130). Consider upward forces as positive, downward as negative.

Then the shearing force at any given cross-section of the beam is the resultant of all the forces acting on the beam from the origin to that cross-section; so that at the series of points where P_1, W_1, W_2, W_3, &c., act in fig. 129, and where W_0, W_1, W_2, W_3, &c., act in fig. 130, it has the series of values,

In Fig. 129.	In Fig. 130.
$F_0 = P_1$;	$-F_0 = W_0$;
$F_1 = P_1 - W_1$;	$-F_1 = W_0 + W_1$;
$F_2 = P_1 - W_1 - W_2$;	$-F_2 = W_0 + W_1 + W_2$;
$F_3 = P_1 - W_1 - W_2 - W_3$	$-F_3 = W_0 + W_1 + W_2 + W_3$;
&c.;	&c.;
and generally,	and generally
$F = P_1 - \Sigma \cdot W$;...(4.)	$-F = \Sigma \cdot W$; ....(5.)

so that the shearing forces at a series of sections can be computed by successive subtractions or successive additions, as the case may be.

For a continuously distributed load, these equations become respectively,

In a beam supported at both ends, $F = P_1 - \int_0^{x'} w\,d\,x$; (6.)

In a beam fixed at one end $-F = \int_0^{x'} w\,d\,x$; .....(7.)

in which expressions, x' denotes the distance from the origin A to the plane of section.

The symbol $-F$ denotes that the shearing force is downward.

The **Greatest Shearing Force** acts in a beam supported at both ends, close to one or other of the points of support, and its value is either P_1 or P_2. In a beam fixed at one end the greatest shearing force on the projecting part acts close to the outer point of support, and its value is equal to the entire load.

In a beam supported at both ends, the **Shearing Force vanishes** at some intermediate section, whose position may be found from equation 4 or equation 6, by making $F = 0$.

STEP III. *To find the Bending Moments at a Series of Sections.*—At the origin A there is no bending moment. Multiply the length of each of the divisions Δx of the longitudinal axis of the beam by the shearing force F, which acts at the outer end of that division; the first of the products so obtained is the bending moment at the inner end of the first division; and by adding to it the other products successively, there are obtained successively the bending moments at the inner ends of the other divisions in succession.*

That is to say,—bending moment,

* This process is substantially the same with that employed by Mr. Herbert Latham, in his work *On Iron Bridges*, to compute the stress in a half-lattice girder.

at the origin $A = 0$;
at the line of action of W_1; $M_1 = F_0 \cdot \Delta x_1$;
" " " W_2; $M_2 = F_0 \cdot \Delta x_1 + F_1 \cdot \Delta x_2$;
" " " W_3; $M_3 = F_0 \cdot \Delta x_1 + F_1 \cdot \Delta x_2 + F_2 \cdot \Delta x_3$;
&c. &c.

$$\text{and generally, } M = \Sigma \cdot F \, \Delta x. \quad \text{(8.)}$$

If the divisions Δx are of equal length, this becomes

$$M = \Delta x \cdot \Sigma F; \quad \text{(9.)}$$

and for a continuously distributed load,

$$M = \int_0^x F \, d x. \quad \text{(10.)}$$

The three preceding equations, 8, 9, and 10, are applicable to beams whether supported at both ends or fixed at one end. By substituting for F in equation 10 its values as given by equations 6 and 7 respectively, we obtain the following results:—

For a beam supported at both ends,

$$M = P_1 x' - \int_0^{x'} \int_0^x w \, d x^2$$

$$= P_1 x' - \int_0^{x'} (x' - x) \, w \, d x; \quad \text{(11.)}$$

For a beam fixed at one end,

$$- M = \int_0^{x'} \int_0^x w \, d x^2 = \int_0^{x'} (x' - x) \, w \, d x; \quad \text{(12.)}$$

in the latter of which equations, the symbol $- M$ denotes that the bending moment acts downwards.

The **Greatest Bending Moment** acts, in a beam fixed at one end, at the outer point of support; and in a beam supported at both ends, at the section where the shearing force vanishes, found, as already stated in Step II., from the equation $F = 0$.

When the load on a beam supported at both ends is symmetrically distributed relatively to its middle section, the Greatest Bending Moment acts at that section; and it is sometimes convenient to assume a point in that section as the origin of co-ordinates.

STEP IV. *To find the effect of combining several Loads on one Beam, whose separate actions are known;*—for each cross-section, the shearing force is the sum of the shearing forces, and the bending moment the sum of the bending moments, which the loads would produce separately

STEP V. *To deduce the Shearing Force and Bending Moment in one Beam from those in another Beam similarly supported and loaded.* —This is done by the aid of the following principles:—

When beams differing in length and in the amounts of the loads upon them are similarly supported, and have their loads similarly distributed, the shearing forces at corresponding sections in them vary as the total loads, and the bending moments as the products of the loads and lengths.

This principle may be expressed by symbols in either of the two following ways:—

First, Let l, l', denote the lengths of two beams, similarly supported; let W, W', denote their total loads, similarly distributed; let F, F', be the shearing forces, and M, M', the bending moments, at sections similarly situated in the two beams; then

$$\mathrm{W} : \mathrm{W}' :: \mathrm{F} : \mathrm{F}', \qquad (13.)$$

$$l\,\mathrm{W} : l'\,\mathrm{W}' :: \mathrm{M} : \mathrm{M}'. \qquad (14.)$$

Secondly, Let k and m be two numerical factors, depending on the way in which a beam is supported, the mode of distribution of its load, and the position of the cross-section under consideration; then

$$\mathrm{F} = k\,\mathrm{W}; \qquad (15.)$$

$$\mathrm{M} = m\,\mathrm{W}\,l. \qquad (16.)$$

Loads upon beams are stated either in pounds, hundredweights, or tons; lengths of beams, in feet, or in inches; and according to the units of load and length employed, the unit of bending moment is a *foot pound,* an *inch pound,* a *foot-hundredweight,* an *inch-hundredweight,* a *foot-ton,* or an *inch-ton,* as the case may be.

As the transverse dimensions of beams are expressed in inches, and their moduli of strength in pounds on the square inch, the most convenient units are, the pound, the inch, and the inch-pound.

The following is a comparison of different unit of bending moment.

Inch-lbs.					
12 =	1 Ft.-lb.				
112 =	$9\frac{1}{3}$ =	1 Inch-cwt.			
1344 =	112 =	12 =	1 Foot-cwt.		
2240 =	$186\frac{2}{3}$ =	20 =	$1\frac{2}{3}$ =	1 Inch-ton.	
26880 =	2240 =	240 =	20 =	12 =	1 Foot-ton.

161. Examples of the Action of a Transverse Load on a Beam.

TABLE OF EXAMPLES.

CASES.	F	x'_1	k	M	x'_0	m
A. BEAMS FIXED AT ONE END.						
I. Loaded at extreme end with W,...............	$-W$	anywhere	-1	$-Wx'$	l	-1
II. Uniform load of intensity $w = W \div l$.	$-wx'$	l	-1	$-\frac{wx'^2}{2}$	l	$-\frac{1}{2}$
III. Uniform load of intensity w and additional load W′ at extreme end,.........	$-W' - wx'$	l	-1	$-W'x' - \frac{wx'^2}{2}$	l	$-\frac{W' + \frac{wl}{2}}{W' + wl}$
B. BEAMS SUPPORTED AT BOTH ENDS.						
IV. Single load W, in the middle;						
half of beam next origin,	$\frac{W}{2}$	0 to $\frac{l}{2}$	$\frac{1}{2}$	$\frac{Wx'}{2}$	$\frac{l}{2}$	$\frac{1}{4}$
farther half, ...	$-\frac{W}{2}$	$\frac{l}{2}$ to l	$-\frac{1}{2}$	$\frac{W(l-x)}{2}$		

TABLE OF EXAMPLES—*continued.*

CASES.	F	x'_1	k	M	x'_0	m
V. Single load W, applied at x'';						
between x'' and origin;	$\frac{l - x''}{l} W$	anywhere	$\frac{l - x''}{l}$	$\frac{x'(l - x'')}{l} W$	x''	$\frac{x''(l - x'')}{l^2}$
beyond x'';	$-\frac{x''}{l} W$	anywhere	$-\frac{x''}{l}$	$\frac{(l - x')x''}{l} W$		
VI. Uniform load of intensity $w = W \div l$,	$w\left(\frac{l}{2} - x'\right)$	0 and l	$\pm\frac{1}{2}$	$\frac{w x'(l - x')}{2}$	$\frac{l}{2}$	$\frac{1}{8}$
VII. Partial load of uniform intensity $w = W \div x''$ from 0 to x'', remainder unloaded;						
between x'' and origin;	$w\left(x'' - \frac{x''^2}{2l} - x'\right)$	0	$1 - \frac{x''}{2l}$	$w\left\{\left(x'' - \frac{x''^2}{2l}\right)x' - \frac{x'^2}{2}\right\}$	$x'' - \frac{x''^2}{2l}$	$\frac{x''}{2l}\left(1 - \frac{x''}{2l}\right)^2$
beyond x''..............	$-\frac{w x''^2}{2l}$	x'' to l	$-\frac{x''}{2l}$	$\frac{w x''^2}{2l}(l - x')$		

—In each example in the preceding table, l denotes, for a beam fixed at one end, the length measured from the outer point of support to the farthest projecting loaded point, and for a beam supported at both ends, the length or *span* between the points of support; W denotes the total load; F and M denote the shearing force and bending moment at any cross-section situated at the distance x' from the origin (which, as in the preceding article, is the point where $M = 0$); F_1 denotes the greatest shearing force; x'_1 the position of the section where it occurs; M_0 the greatest bending moment; x'_0 the position of the cross-section where it occurs; k, m, the two factors employed in equations 15 and 16, for the sections of greatest shearing stress and greatest bending moment respectively; that is to say,

$$k = F_1 \div W;\ m = M_0 \div W\,l. \qquad (1.)$$

To transform the expressions in the preceding table, Cases IV. to VII., which are suited for co-ordinates measured from one point of support of a beam supported at both ends, into expressions suited for co-ordinates measured from the middle of the beam, let c be the *half-span*, and substitute $2\,c$ for l, $c - x$ for x', and $c + x$ for $l - x'$, throughout the whole of that part of the table.

In the following example, a beam supported at both ends is supposed to be loaded at a series of detached points, which divide the length of the beam into N equal divisions, so that the length of one of those divisions is $l \div N$. The origin of co-ordinates being at a point of support, the plane of section in each example is supposed to be immediately *beyond* the n^{th} division from that point.

Case.		x'_1	k	M	x'_0	m
VIII. Each intermediate point loaded with w; total load $(N-1)\,w$ .... l	$\left(\frac{N-1}{2} - n\right) w$	0 to $\frac{l}{N}$ or $\frac{N-1}{N}l$ to l		$\frac{n(N-n)}{2N}\,l\,w$	$\frac{l}{2}$ (N even), $\frac{N-1}{2}\frac{l}{N}$ to $\frac{N+1}{2}\frac{l}{N}$ (N odd)	(N even) $\frac{1}{8}$ (N odd) $\frac{N+1}{8N}$

Case IX. *Travelling Load.*—A beam of the span l is supported at the two ends; a permanent load of the uniform intensity of w lbs. per lineal foot is distributed over it. An additional load, such as the weight of a railway train, of w' lbs. per lineal foot, gradually rolls on to the beam from one end, covering it at last from end to

end, and then rolls off again at the other end. It is required to find the greatest shearing force at any given section, and also the greatest bending moment.

The *Greatest Shearing Force* at a given cross-section occurs when the longer of the two segments into which it divides the beam is loaded with the travelling load as well as with the permanent load, and the shorter loaded with the permanent load only. Let F' denote that force, and x the distance of the section in question from the nearer end of the beam; then

$$F' = w\left(\frac{l}{2} - x\right) + \frac{w'(l-x)^2}{2l} \ldots\ldots\ldots\ldots\ldots (2.)$$

This formula may be rendered somewhat more symmetrical by taking the *middle* of the beam, instead of one of the ends, as origin of co-ordinates. Let x', then, denote the distance of the section in question from the middle of the beam, and $c = l \div 2$, its half-span; then $x = c - x'$, and

$$F' \qquad + \frac{w'(c+x')^2}{4c}. \qquad (3.)$$

The *Greatest Bending Moment* at each section occurs when the travelling load extends over the whole length of the beam; so that in this respect no difference exists between the present case and case VI. of the preceding table; that is to say,

$$M = \frac{(w+w')\,x\,(l-x)}{2} = \frac{(w+w')(c^2-x'^2)}{2} \ldots. (4.)$$

The preceding principles are represented graphically by the following construction. In fig. 131, let A B represent the span of the beam; bisect it in C. Through A draw D A E perpendicular to A B, and take A D to represent *half the greatest permanent load*, that is, A D $= w\,l \div 2$, and A E to represent on the same scale *half the greatest travelling load*, that is, A E $= w'\,l \div 2$. Draw the straight line C D; also the parabola B L E, having its vertex at B, touching A B. About C describe the semicircle A F B, and consider C F^2, or the square of the radius of that semicircle perpendicular to A B, as representing the *greatest bending moment at the middle of the beam*, or $(w + w')\,l^2 \div 8$. Let G represent the position of any cross-section of the beam. Draw H G I K perpendicular to A B; then H I will represent the greatest

Fig. 131.

shearing force at that section, and G K² the greatest bending moment.

The ordinate C L of the parabola at the middle of the span, being one-fourth of A E, represents the greatest shearing force in the middle of the beam, which is one-eighth of the greatest travelling load, or $w' l \div 8$.

CASE X.—If a beam has *equal and opposite couples* applied to its two ends,—for example, if the beam in fig. 132 has the couple of equal and opposite forces P_1 applied at A and B, and the couple of equal and opposite forces P_2 at C and D, and if the opposite moments,

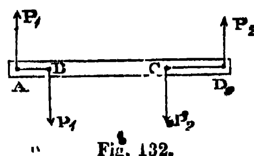


Fig. 132.

$$P_1 \cdot A B = P_2 \cdot C D = M, \quad \ldots\ldots\ldots\ldots\ldots\ldots (5.)$$

are equal, then each of the endmost divisions of the beam, A B and C D, is in the condition of a beam fixed at one end and loaded at the other (Case I.); and the middle division B C is subject to the *uniform moment of flexure* M, and to no shearing force.

162. **Resistance to Cross-Breaking—Transverse Strength.** (*A. M.*, 293 to 295, 309, 310.)—The bending moment at each cross-section of a beam bends the beam so as to make any originally plane layer of the beam, perpendicular to the direction of the load, become concave in the direction towards which the moment acts, and convex in the opposite direction. Thus, fig. 133 represents a side view of a short portion of a beam supported at the ends and loaded at intermediate points; C C′ is a layer, originally plane, and perpendicular to the direction of the load, which is now bent so as to become concave above and convex below.

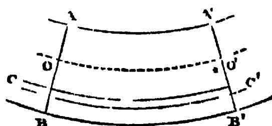


Fig 133.

The layers at and near the concave side of the beam A A′ are shortened, and the layers near the convex side B B lengthened, by the bending action of the load. There is one intermediate surface O O′ which is neither lengthened nor shortened; it is called the "*neutral surface*." The particles at that surface are not necessarily, however, in a state devoid of strain; for, in common with the other particles of the beam, they are compressed and extended in a pair of diagonal directions, making angles of 45° with the surface, by the shearing action of the load, when such action exists.

The condition of the particles of a beam, produced by the combined bending and shearing actions of the load, is illustrated by fig. 134, which represents a vertical longitudinal section of a rect-

angular beam, supported at the ends, and loaded at intermediate points. It is covered with a net-work consisting of two sets of curves cutting each other at right angles. The curves convex upwards are *lines of direct thrust;* those convex down-wards are *lines of direct tension.* A pair of tangents to the pair of curves which traverse any particle, are the *axes of stress* of that particle. (See Art. 108, p. 167.) The *neutral surface* is cut by both sets of curves at angles of 45°. At that vertical section of the beam where the shearing force vanishes, and the bending moment is greatest, both sets of curves become horizontal.



Fig. 134.

Except in cases to be afterwards specified, it is unnecessary to consider the shearing action of the load on a beam.

When a beam breaks under the bending action of its load, it gives way either by the crushing of the compressed side, A A′, or by the tearing of the stretched side, B B′.

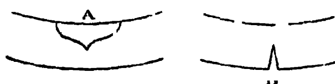


Fig. 135.

In fig. 135, A represents a beam of a granular material, like cast iron, giving way by the crushing of the compressed side, out of which a sort of wedge is forced. B represents a beam giving way by the tearing asunder of the stretched side. (See p. 806.)

The *resistance* of a beam to bending and cross-breaking at any given cross-section is the moment of the couple, consisting of the thrust along the longitudinally-compressed layers, and the equal and opposite tension along the longitudinally-stretched layers.

It has been found by experiment, that in most cases which occur in practice, the longitudinal stress of the layers of a beam may, without material error, be assumed to be *uniformly-varying* (Article 106, p. 163), its intensity being simply proportional to the distance of the layer from the neutral surface.

Let fig. 136 represent a cross-section of a beam (such as that represented in fig. 133), A the compressed side, B the extended side, C any layer and O O the *neutral axis* of the section, being the line in which it is cut by the neutral surface. Let p denote the intensity of the stress along the layer C, and y the distance of that layer from the neutral axis; because the stress is uniformly varying, $p \div y$ is a constant quantity. Let that constant be denoted for the present by a.

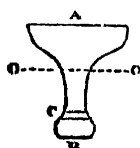


Fig. 136.

Let z be the breadth of the layer C, and $d\,y$ its thickness;
Then the amount of the stress along it is

$$p\,z\,d\,y = a\,y\,z\,d\,y;$$

the amount of the stress along all the layers at the given cross-section is

$$a \int y\,z\,d\,y;$$

and this amount must be nothing,—in other words, the total thrust and total tension at the cross-section must be equal,—because the forces applied to the beam are wholly transverse; from which it follows, that

$$\int y\,z\,d\,y = 0, \quad \text{.........................(1.)}$$

and the *neutral axis traverses the centre of gravity of the cross-section.* This principle enables the neutral axis to be found by the aid of the methods explained in Articles 104 and 105, pp. 152 to 158; it being borne in mind that the process of finding the centre of gravity of a given plane figure is the same with that of finding the centre of gravity of a homogeneous uniformly thick flat plate of that figure.

To find the greatest value of the constant $p \div y$ consistent with the strength of the beam at the given cross-section, let y_a be the distance of the compressed side, and y_b that of the extended side from the neutral axis; f_a the greatest thrust, and f_b the greatest tension, which the material can bear in the form of a beam; compute $f_a \div y_a$, and $f_b \div y_b$, and adopt the *less* of those two quantities as the value of $p \div y$, which may now be denoted by $f \div y_1$; f being f_a or f_b, and y_1 being y_a or y_b, according as the beam is liable to give way by crushing or by tearing.

The moment relatively to the neutral axis, of the stress exerted along any given layer of the cross-section, is

$$y\,p\,z\,d\,y = \frac{f}{y_1}\,y^2\,z\,d\,y;$$

and the sum of all such moments, being the

Moment of Resistance of the given cross-section of the beam to breaking across, is given by the formula,

$$M_0 = \int p\,y\,z\,d\,y = \frac{f}{y_1} \int y^2\,z\,d\,y; \quad \text{...............(2.)}$$

or making $\int y^2 z \, d y = \mathrm{I}$,

$$\mathrm{M}_0 = \frac{f\mathrm{I}}{y_1} \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (2\ \mathrm{A.})$$

When the *breaking* load is in question, the co-efficient f is what is called the MODULUS OF RUPTURE of the material. It does not always agree with the resistance of the same material to direct crushing or direct tearing, but has a special value, which can be found by experiments on cross-breaking only. One of the causes of this phenomenon is probably the fact, already stated in Article 154, p. 229, that the resistance of a material to a direct stress is increased by preventing or diminishing the alteration of its transverse dimensions; and another cause may be the fact, that the strength of masses of metal, especially when cast, is greater in the external layer, or *skin*, than in the interior of the mass. When a bar is directly torn asunder, the strength indicated is that of the weakest part of the mass, which is in the centre; when it is broken across, the strength indicated is that either of the skin, which is the strongest part, or of some part near the skin.

When the *proof load* or *working load* is in question, the co-efficient f is the modulus of rupture divided by a suitable *factor of safety*, as to which see Article 143, p. 222.

The factor denoted by I in the preceding equation is what is conventionally called the "*moment of inertia*" of the cross-section of the beam. For sections whose figures are similar, or are parallel projections of each other, the moments of inertia are to each other as the breadths, and as the cubes of the depths of the sections; and the values of y_1 are as the depths. If, therefore, b be the breadth and h the depth of the rectangle circumscribing the cross-section of a given beam at the point where the moment of flexure is greatest, we may put

$$\mathrm{I} = n' b h^3 \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (3.)$$

$$y = m' h \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (4.)$$

n' and m' being numerical factors depending on the form of section; and making $n' \div m' = n$, the moment of resistance may be thus expressed,—

$$\mathrm{M}_0 = n f b h^2 \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (5.)$$

Hence it appears, that the *resistances of similar cross-sections to cross-breaking are as their breadths and as the squares of their depths.*

The relation between the load and the dimensions of a beam is found by equating the value of the greatest bending moment in terms of the load and length of the beam, as given in Article 160, equations 10, 11, 12, and 16, pp. 243, 244, to the value of the moment of resistance of the beam, at the cross-section where that greatest bending moment acts, as given in equation 5 of this Article; that is to say,

$$M_0 = m W l = n f b h^2 \dots\dots\dots\dots\dots\dots (6.)$$

m being the co-efficient depending on the mode of distribution of the load, as defined by equation 1 of Article 161, p. 247, and given for particular cases in the tables and examples of Article 161.

In using the above equation 6, it is to be understood that the same unit of measure is to be employed for all the dimensions of the beam; and inasmuch as the values of the "modulus of rupture" f given in tables are generally stated in *pounds on the square inch*, so that the *inch* is the proper unit for the transverse dimensions b and h, the length or span l ought to be expressed in inches also, so that the bending moment will be computed in *inch-pounds*.

In finding the value of the moment of inertia I of cross-sections of complex figure, the following rules are useful.

If a complex cross-section is made up of a number of simple figures, conceive the centre of gravity of each of those figures to be traversed by a neutral axis parallel to the neutral axis of the whole section. Find the moment of inertia of each of the component figures relatively to its own neutral axis; multiply its area by the square of the distance between its own neutral axis and the neutral axis of the whole section; and add together all the results so found, for the moment of inertia of the whole section. To express this in symbols, let A′ be the area of any one of the component figures, y' the distance of its neutral axis from the neutral axis of the whole section, I′ its moment of inertia relatively to its own neutral axis; then the moment of inertia of the whole section is

$$I = \Sigma \cdot I' + \Sigma \cdot y'^2 A'. \dots\dots\dots\dots\dots\dots (7.)$$

When the figure of the cross-section can be made by *taking away* one simpler figure from another, so that the centre of gravity of the whole figure is found *by subtraction* (as in p. 154), both the area and the moment of inertia of the subtracted figure are to be considered as negative, and so treated, in making use of equation 7.

163. Examples of Moment of Resistance.—The following table contains examples of the values of the factors n', m', and n, of equations 3, 4, and 5:—

FORM OF CROSS SECTIONS.	$n' = \frac{I}{bh^3}$	$m' = \frac{y_1}{h}$	$n = \frac{M_0}{fbh^2}$
I. Rectangle bh, (including square)	$\frac{1}{12}$	$\frac{1}{2}$	$\frac{1}{6}$
II. Ellipse— Vertical axis h, Horizontal axis b, (including circle)	$= 0{\cdot}0491$	$\frac{1}{2}$	$\frac{\pi}{32} = \frac{1}{10{\cdot}2} = 0{\cdot}0982$
III. Hollow rectangle, $bh - b'h'$; also I-formed section, where b' is the sum of the breadths of the lateral hollows,	$\frac{1}{12}\left(1 - \frac{b'h'^3}{bh^3}\right)$	$\frac{1}{2}$	$\frac{1}{6}\left(1 - \frac{b'h'^3}{bh^3}\right)$
IV. Hollow square— $h^2 - h'^2$	$\frac{1}{12}\left(1 - \frac{h'^4}{h^4}\right)$	$\frac{1}{2}$	$\frac{1}{6}\left(1 - \frac{h'^4}{h^4}\right)$
V. Hollow ellipse,	$\frac{1}{20{\cdot}4}\left(1 - \frac{b'h'^3}{bh^3}\right)$	$\frac{1}{2}$	$\frac{1}{10{\cdot}2}\left(1 - \frac{b'h'^3}{bh^3}\right)$
VI. Hollow circle,	$\frac{1}{20{\cdot}4}\left(1 - \frac{h'^4}{h^4}\right)$	$\frac{1}{2}$	$\frac{1}{10{\cdot}2}\left(1 - \frac{h'^4}{h^4}\right)$
VII. Isosceles triangle; base b, height h; y_1 measured from summit,	$\frac{1}{36}$	$\frac{2}{3}$	$\frac{1}{24}$

The following examples are not well suited for introduction into the tables:—

EXAMPLE VIII.—T-formed section.

	Areas.	Depths.
Flange or table, ...	A_1	h_1
Vertical web, ...	A_2	h_2

$$\text{Totals, } A_1 + A_2 = A;\ h_1 + h_2 = h.$$

Exact Solution.—Distance of the neutral axis from the edge of the vertical web,—

$$y_1 = \frac{h}{2} + \frac{h_2 A_1 - h_1 A_2}{2 A};$$

Moment of Inertia of whole section,—

$$I = \frac{A_1 h_1^2 + A_2 h_2^2}{12} + \frac{A_1 A_2 (h_1 + h_2)^2}{4 A}; \quad \Bigg\} \ldots\ldots(1.)$$

Moment of Resistance, as before,—

$$M_0 = \frac{f I}{y_1}$$

Approximate Solution.—When h_1 is small compared with h_2, make $h' = h_2 + \frac{1}{2} h_1$; then, the following are nearly correct:—

$$y_1 = \frac{h'}{2}\left(1 + \frac{A_1}{A}\right); \quad I = h'^2\left(\frac{A_2}{12} + \frac{A_1 A_2}{4A}\right);$$

$$M_0 = \frac{fI}{y_1} = \frac{fh'}{6} \cdot \frac{(A + 3 A_1) A_2}{A + A_1} \quad \ldots\ldots\ldots(2.)$$

$$= \frac{fh'}{6} \cdot \frac{A_2 (A_2 + 4 A_1)}{A_2 + 2 A_1}.$$

EXAMPLE IX.—Double T-formed section. The beam is assumed to give way by the tearing of the lower flange or table B. Let the areas and depths of the parts of which the section consists be denoted as follows:—

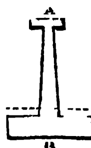


Fig. 37.

	Areas.	Depths.
Upper flange or table	A_1,	h_1.
Vertical web,	A_2,	h_2.
Lower flange or table,	A_3,	„
Totals,......$A_1 + A_2 + A_3 = A$,	$h_1 + h_2 + h_3 = h$.	

Exact Solution.—The height of the neutral axis above the lower side of this section is

$$y_0 = \frac{h}{2} - \frac{(h_2 + h_1) A_3 - (h_3 + h_2) A_1 - (h_3 - h_1) A_2}{2 A}.$$

The Moment of Inertia of the section,—

$$I = \frac{A_1 h_1^2 + A_2 h_2^2 + A_3 h_3^2}{12} + \frac{1}{4 A}\Big\{ A_1 A_3 (h_1 + h_3 + 2 h_2)^2 + A_1 A_2 (h_1 + h_2)^2 + A_2 A_3 (h_2 + h_3)^2 \Big\}; \quad (3.)$$

and the Moment of Resistance,—

$$M_0 = \frac{f_b I}{y_0};$$

Approximate Solution.—When h_1 and h_3 are small compared with h_2, make

$$h' = h_2 + \frac{h_1 + h_3}{2};$$

then the following formulæ are nearly correct:—

$$y_b = \frac{h'}{2}\left(1 - \frac{A_3 - A_1}{A}\right) = \frac{h'}{2}\cdot\frac{A_2 + 2A_1}{A}; \qquad (4.)$$

$$I = h'^2 \left\{ \frac{A_2}{12} + \frac{A_2 A_3 + A_1 A_2 + 4 A_1 A_3}{4 A} \right\};$$

$$M_0 = \frac{f_b I}{y_b} = \frac{f_b h'}{6}\cdot\frac{A_2 (A_2 + 4 A_1 + 4 A_3) + 12 A_1 A_3}{A_2 + 2 A_1};$$

Another Approximate Solution, when A_2 is very small, or when there is an open frame instead of a vertical web—

$$y_b = \frac{h' A_1}{A};\ I = \frac{h'^2 \cdot A_1 A_3}{A};\ M_0 = f_b h' A_3. \quad \ldots\ldots\ldots\ldots(5.)$$

164. **Cross-section of Equal Strength.**—The use of the T-shaped and double-T-shaped cross-sections mentioned in the last Article, is to economize any material whose resistances to cross-breaking by crushing and by tearing are different, by so adjusting the position of the neutral axis, that the tendencies of the beam to break across by crushing and by tearing shall be as nearly as possible equal. The following are the rules for effecting that adjustment:—

Let f_a be the modulus of rupture by crushing;
„ f_b „ „ „ by tearing;
„ y_a the distance from the neutral axis to the compressed side;
„ y_b „ „ „ extended side;
$y_a + y_b = h$ being the depth of the beam;

then the neutral axis should be so placed as to divide that depth in the following proportion:—

$$\left.\begin{array}{l} f_a + f_b : f_a : f_b \\ h : y_a : y_b \end{array}\right\} \qquad (1.)$$

Let A_2, as before, denote the area of the cross-section of the vertical web of a beam, *measured from centre to centre* of the top and bottom flanges; A_1 the area of the compressed flange, A_3 that of the extended flange.

The following solutions are to the same degree of approximation with those in equations 2 and 4 of Article 163.

CASE I. *f_a greater than f_b, beam T-shaped.*—Here the flange is required at the stretched side of the girder; and its area must be as follows:—

$$A_3 = \frac{f_a - f_b}{2 f_b} A_2 \text{.............................(2.)}$$

The Moment of Resistance of this form of cross-section is (see Article 163, equation 2, p. 255)—

$$M_0 = \frac{(2 f_a - f_b) h' A_2}{6} \text{........................(3.)}$$

CASE II *f_a greater than f_b, beam double T-shaped.*—The area of the *compressed* flange being A_1, that of the *stretched* flange should be as follows:—

$$A_3 = \frac{f_a}{f_b} A_1 + \frac{f_a - f_b}{2 f_b} A_2; \text{......................(4.)}$$

when the Moment of Resistance will become

$$M_0 = h' \left\{ f_a A_1 + (2 f_a - f_b) \frac{A_2}{6} \right\}$$

$$= h' \left\{ f_b A_3 - (f_a - 2 f_b) \frac{A_2}{6} \right\} \text{ ....(5.)}$$

In designing a beam to bear a given bending moment, the depth h' and area A_2 of the vertical web are to be fixed by considerations of practical convenience, when equation 5 will enable the areas of either or both of the flanges to be computed.

Example.—Suppose that for a certain sort of cast iron,

f_a = 80,000 lbs. in the square inch;
f_b = 20,000 „ „ „

so that in a well proportioned section,

$$5 : 4 : 1 :: h : y_a : y_b.$$

Then in a T-shaped section,

$$A_3 = \frac{4 - 1}{2} A_2 = \frac{3}{2} A_2; \text{ and}$$

$$M_0 = \frac{140,000\ h'\ A_2}{6}; \qquad \text{(6.)}$$

and in a double T-shaped section,

$$A_3 = 4 A_1 + \frac{3}{2} A_2;\text{*} \text{ and}$$

* This result agrees nearly with the proportions which Mr. Hodgkinson found experimentally to be the best for cast iron beams.

$$M_0 = h' \left\{ 80{,}000\, A_1 + 140{,}000 \frac{A_2}{6} \right\}$$

$$= h' \left\{ 20{,}000\; A_3 - 40{,}000 \frac{A_2}{6} \right\} \ldots\ldots\ldots\ldots\ldots\ldots(7.)$$

CASE III. *f_a less than f_b, beam T-shaped.*—Here the flange is required at the *compressed* side of the beam, and its area should be,

$$A_1 = \frac{f_b - f_a}{2 f_a} \cdot A_2. \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots(8.)$$

The Moment of Resistance of this cross-section is

$$M_0 = \frac{(2 f_b - f_a)\, h'\, A_2}{6} \ldots\ldots\ldots\ldots\ldots\ldots\ldots(9.)$$

CASE IV. *f_a less than f_b, beam double T-shaped.*—The area of the *stretched* flange being A_3, that of the *compressed* flange should be as follows.—

$$A_1 = \frac{f_b}{f_a} A_3 + \frac{f_b - f_a}{2 f_a} A_2; \ldots\ldots\ldots\ldots\ldots\ldots(10.)$$

when the Moment of Resistance will become

$$M_0 = h' \left\{ f_b\, A_3 + (2 f_b - f_a) \frac{A_2}{6} \right\}$$

$$= h' \left\{ f_a\, A_1 + (2 f_a - f_b) \frac{A_2}{6} \right\} \ldots\ldots\ldots\ldots\ldots(11.)$$

In designing a beam to bear a given bending moment, the depth h' and area A_2 of the vertical web are to be fixed by considerations of practical convenience, when equation 11 will enable the area of either or both of the flanges to be computed.

Example.—Suppose that for a certain sort of wrought iron,

$f_a = 36{,}000$ lbs. on the square inch.
$f_b = 60{,}000$ „ „ „ „

so that in a well-proportioned section

$$8 : 3 : 5 :: h : y_a : y_b.$$

Then in a T-shaped section,

$$A_1 = \frac{5-3}{\ } A_0 = \frac{A_2}{\ };$$

$$M_0 = \frac{84{,}000\, h' A_2}{6}; \qquad (12.)$$

and in a double T-shaped section

$$A_1 = \frac{5}{3} A_3 + \frac{1}{3} A_2;$$

$$M_0 = h' \left\{ 60{,}000\, A_3 + 84{,}000 \frac{A_2}{6} \right\}$$

$$= h' \left\{ 36{,}000\, A_1 + 12{,}000 \frac{A_2}{6} \right\}.* \ldots\ldots\ldots\ldots (13.)$$

165. **Longitudinal Sections of Uniform Strength for Beams** (*A. M.*, 299) are those in which the dimensions of the cross-section are varied in such a manner that its ultimate moment of resistance bears at each point of the beam the same proportion to the bending moment of the load. That moment of resistance, for figures of the same kind, being proportional to the breadth and to the square of the depth, can be varied either by varying the breadth, the depth, or both. The law

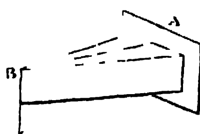


Fig. 138.

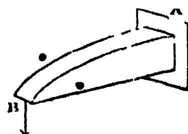


Fig. 139.

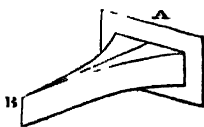


Fig. 140.

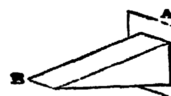


Fig. 141.

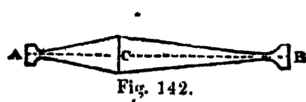


Fig. 142.

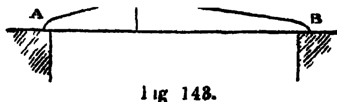


Fig. 143.

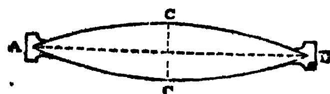


Fig. 144.

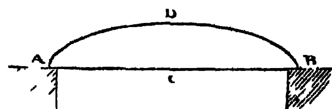


Fig. 145.

* The first theoretical solution of Cases II. and IV. was contained in a paper by Mr. Callcott Reilly, read to the British Association at Oxford in 1860.

of variation depends upon the mode of variation of the moment of flexure of the beam from point to point, and this depends on the distribution of the load and of the supporting forces, in a way which has been exemplified in Articles 160 and 161. When the depth of the beam is made uniform, and the breadth varied, the vertical longitudinal section is rectangular, and the plan is of a figure depending on the mode of variation of the breadth. When the breadth of the beam is made uniform, and the depth varied, the plan is rectangular, and the vertical longitudinal section is of a figure depending on the mode of variation of the depth. The following table gives examples of the results of those principles:—

Mode of Loading and Supporting.	bh^2 proportional to	Depth h constant; Figure of Plan.	Breadth b constant; Figure of Vertical Longitudinal Section
I. (Figs. 138, 139) Fixed at A. loaded at B, ...	Distance from B.	Triangle, apex at B, fig. 138.	Parabola, vertex at B, fig. 139.
II. (Figs. 140, 141). Fixed at A, uniformly loaded,...	Square of distance from B.	Pair of parabolas, vertices touching each other at B, fig. 140.	Triangle, apex at B, fig. 141.
III. (Figs. 142, 143) Supported at A and B, loaded at C, ...	Distance from adjacent point of support.	Pair of triangles, common base at C, apices at A and B, fig. 142.	Pair of parabolas, vertices at A and B, meeting at C, fig. 143.
IV. (Figs. 144, 145) Supported at A and B, uniformly loaded, ...	Product of distances from points of support.	Pair of parabolas, vertices at C, C, in middle of beam; common base A B, fig. 144.	Ellipse A D B, fig. 145.

The formulæ for a *constant depth* are applicable, approximately, to the breadths of the flanges of the T-shaped and double T-shaped girders, described in Article 164. In applying the principles of this Article, it is to be borne in mind, that the *shearing force* has not yet been taken into account; and that, consequently, the figures described in the above table require, at and near the places where they taper to edges, some additional material to enable them to withstand that force. In figs. 142 and 144, such additional

material is shown, disposed in the form of projections or palms at the points of support, which serve both to resist the shearing force, and to give lateral steadiness to the beams. As to the greatest intensity of the shearing stress, see Article 168.

166. **Modulus of Rupture of Cast Iron Beams.** (*A. M.*, 297.)—Mr. William Henry Barlow, in a paper read to the Royal Society (see *Phil. Trans.*, 1855), showed that the modulus of rupture of cast iron beams has various values, ranging from the mere direct tenacity of the iron up to about two-and a-quarter times that tenacity, according to the figure of the cross-section of the beam. This was proved by experiments on beams, which were, in some cases, of a solid rectangular section, and in other cases, of an open work rectangular section. So far as those experiments went, they were in accordance with the following empirical formula:—

$$f = f_0 + f' \cdot \frac{H}{h}; \quad \text{.......................(1.)}$$

where f is the modulus of rupture of the beam in question; f_0, the direct tenacity of the iron of which it is made; f', a co-efficient determined empirically; and $\frac{H}{h}$, the ratio which the *depth of solid metal* H in the cross section of the beam bears to the *total depth of section h*. The following were the values of the constants for the cast iron experimented on:—

$$\left.\begin{array}{l} \text{Direct tenacity, } f_0 = 18{,}750 \text{ lbs. per square inch;} \\ \qquad\qquad\quad f' = 23{,}000 \text{ lbs. per square inch;} \\ \qquad\qquad\qquad = 1\tfrac{1}{4} f_0 \text{ nearly.} \end{array}\right\} \text{............(2.)}$$

Mr. Barlow afterwards made further experiments on cast iron beams of various forms of section, and also experiments on wrought iron beams, showing, though not so conclusively, variations in the modulus of rupture of wrought iron analogous to those which have been proved to exist in the case of cast iron.

167. **Allowance for Weight of Beam—Limiting Length of Beam.** (*A. M.*, 314, 315.)—When a beam is of great span, its own weight may bear a proportion to the load which it has to carry, sufficiently great to require to be taken into account in determining the dimensions of the beam. The following is the process to be performed for that purpose, when the load is uniformly distributed, and the beam of uniform cross-section. Let W′ be the external working load, s_1 its factor of safety, s_2 a factor of safety suited to a steady load, like the weight of the beam.

Let b' denote the breadth of any part of the beam, as computed by considering the *external breaking load alone*, s_1 W′. Compute

the weight of the beam from that *provisional* breadth, and let it be denoted by B′. Then $\frac{s_1 W'}{s_1 W' - s_2 B'}$ is the proportion in which the *gross* breaking load exceeds the external part of that load. Consequently, if for the *provisional* breadth b' there be substituted the *exact* breadth,

$$b = \frac{b' s_1 W'}{s_1 W' - s_2 B'}, \qquad (1.)$$

the beam will now be strong enough to bear both the proposed external load W′, and its own weight, which will now be

$$B = \frac{B' s_1 W'}{s_1 W' - s_2 B'}; \qquad (2.)$$

and the true gross breaking load will be

$$W = s_1 W' + s_2 B = \frac{s_1 W'^2}{s_1 W' - s_2 B'}. \qquad (3.)$$

As the factor of safety for a steady load is in general one-half of that for a moving load, s_1 may be made $= 2 s_2$; in which case the preceding formulæ become

$$b = \frac{2 b' W'}{2 W' - B'}; \qquad (4.)$$

$$B = \frac{2 B' W'}{2 W' - B'}; \qquad (5.)$$

$$W = \frac{2 s_1 W'^2}{2 W' - B'}. \qquad (6.)$$

In all these formulæ, both the external load and the weight of the beam are treated as if uniformly distributed—a supposition which is sometimes exact, and always sufficiently near the truth for the purposes of the present Article.

The gross load of beams of similar figures and proportions, varying as the breadth and square of the depth directly, and inversely as the length, is proportional to the square of a given linear dimension. The weights of such beams are proportional to the cubes of corresponding linear dimensions. Hence the weight increases at a faster rate than the gross load; and for each particular figure of a beam of a given material and proportion of its dimensions, there must be a certain size at which the beam will bear its own weight only, without any additional load.

To reduce this to calculation, let the uniformly distributed gross breaking load of a beam of a given figure be expressed as follows:—

$$W = s_1 W' + s_2 B = \frac{8 n f h A}{l}; \qquad (7.)$$

l, h, and A being the length, depth, and sectional area of the beam, f the modulus of rupture, and n a factor depending on the form of cross-section. The weight of the beam will be expressed by

$$B = k w' l A; \qquad (8.)$$

w' being the weight of an unit of volume of the material, and k a factor depending on the figure of the beam. Then the ratio of the weight of the beam multiplied by its proper factor of safety to the gross breaking load is

$$\frac{s_2 B}{W} = \frac{s_2 k w' l^2}{8 n f h}; \qquad (9.)$$

which increases in the simple ratio of the length, if the proportion $l \div h$ is fixed. When this is the case, the length L of a beam, whose weight (treated as uniformly distributed) is its working load, is given by the condition $s_2 B = W$; that is,

$$L = \frac{8 n f h}{s_2 k w' l}; \qquad (10.)$$

This *limiting length* having once been determined for a given class of beams, may be used to compute the ratios of the gross breaking load, weight of the beam, and external working load to each other, for a beam of the given class, and of any smaller length, l, according to the following proportional equation:—

$$L : \frac{l}{s_2} : \frac{L - l}{s_1} :: W : B : W'; \qquad (11.)$$

In all the following examples, the factors of safety employed are $s_1 = 6$; $s_2 = 3$:—

EXAMPLE I.—Let the beams in question be plain *rectangular cast iron beams*, so that $n = \frac{1}{6}$, $k = 1$, $w' = 0{\cdot}257$ lb. per cubic inch; let $f = 40{,}000$ lbs. per square inch; and let $\frac{h}{l} = \frac{1}{15}$. Then

$$L = 4{,}612 \text{ inches} = 384 \text{ feet nearly}; \qquad (12.)$$

also, l being expressed in inches—

$$4{,}612 : \frac{l}{3} : \frac{4612 - l}{6} :: W : B : W'. \qquad (13.)$$

EXAMPLE II.—Cast iron beam of uniform single T-shaped section of equal strength (as in Article 164, Case I., p. 257).

$$A_3 = \frac{3}{2} A_2; \therefore A = \frac{5}{2} A_2 \text{ and } A_2 = \frac{2}{5} A.$$

$$M_0 = \frac{W l}{8} = \frac{7 f_a h' A_2}{24} = \frac{14 f_a h' A}{120}; \therefore W = \frac{14 f_a h' A}{15 l}.$$

As before, $B = 0{\cdot}257\ l\ A$;

Let $f_a = 80{,}000$ lbs on the square inch; $\frac{h'}{l} = \frac{1}{15}$; then

$$\left.\begin{array}{l} \frac{s_2 B}{W} = \frac{l \text{ in inches}}{6{,}456}, \text{ and} \\ L = 6{,}456 \text{ inches} = 538 \text{ feet}; \end{array}\right\} \ldots\ldots\ldots\ldots(14.)$$

also,

$$6{,}456 : \frac{l}{3} : \frac{6{,}456 - l}{6} : : W : B : W'. \ldots\ldots\ldots\ldots(15.)$$

EXAMPLE III.—Cast iron beam of uniform double T-shaped section of equal strength (as in Article 164, Case II., p. 257).

$$A_3 = \frac{3}{2} A_2 + 4 A_1; \therefore A = \frac{5}{2} A_2 + 5 A_1.$$

$$M_0 = \frac{W l}{8} = f_a h \left(\frac{7}{24} A_2 + A_1\right); \therefore W = \frac{f_a h'}{l} \left(\frac{7}{3} A_2 + 8 A_1\right).$$

In order to obtain a definite result, some proportion must be assumed between the area of the upper flange A_1, and that of the vertical web A_2. For the sake of illustration, let $A_1 = A_2 \div 2$; which proportion is not unusual in practice. Then

$$A = 5 A_2 = 10 A_1 = \frac{10}{7} A_3; \text{ and}$$

$$W = \frac{f_a h' A}{l} \left(\frac{7}{15} + \frac{4}{5}\right) = \frac{19}{15} \cdot \frac{f_a h' A}{l}.$$

As before, $B = 0{\cdot}257\ l\ A$. Let $f_a = 80{,}000$ lb. on the square inch; $\frac{h'}{l} = \frac{1}{15}$; then

$$\left.\begin{array}{l} \frac{s_2 B}{W} = \frac{l \text{ in inches}}{8{,}762}; \text{ and} \\ L = 8{,}762 \text{ inches} = 730 \text{ feet nearly}; \end{array}\right\} \ldots\ldots\ldots(16.)$$

also,

$$8{,}762 : \frac{l}{3} : \frac{8{,}762 - l}{6} : : \mathrm{W} : \mathrm{B} : \mathrm{W}' \ldots\ldots\ldots (17.)$$

EXAMPLE IV.—Wrought iron beam of uniform single T-shaped section of equal strength (as in Article 164, Case III., p. 258).

$$\mathrm{A}_1 = \frac{1}{3} \mathrm{A}_2 ; \therefore \mathrm{A} = \frac{4}{3} \mathrm{A}_2 \text{ and } \mathrm{A}_2 = \frac{3}{4} \mathrm{A}_1$$

$$\mathrm{M}_0 = \frac{\mathrm{W}\, l}{8} = \frac{7 f_b h' \mathrm{A}_2}{30} = \frac{7 f_b h' \mathrm{A}}{40} ; \therefore \mathrm{W} = \frac{7 f_b h' \mathrm{A}}{5 l}.$$

In this case, $\mathrm{B} = 0{\cdot}277\, l\, \mathrm{A} = \frac{5}{18} l\, \mathrm{A}$.

Let $f_b = 60{,}000$ lbs. on the square inch; $\frac{h'}{l} = \frac{1}{15}$; then

$$\left.\begin{aligned} \frac{s_2\, \mathrm{B}}{\mathrm{W}} &= \frac{l \text{ in inches}}{6{,}720}; \text{ and} \\ \mathrm{L} = 6{,}720 \text{ inches} &= 560 \text{ feet nearly}, \end{aligned}\right\} \ldots\ldots (18.)$$

$$6{,}720 : \frac{l}{3} : \frac{6{,}720 - l}{6} : : \mathrm{W} : \mathrm{B} : \mathrm{W}' \ldots\ldots\ldots (19.)$$

EXAMPLE V.—Wrought iron beam of uniform double T-shaped section of equal strength (as in Article 164, Case IV., p. 258).

$$\mathrm{A}_1 = \frac{1}{3} \mathrm{A}_2 + \frac{5}{3} \mathrm{A}_3 ; \therefore \mathrm{A} = \frac{4}{3} \mathrm{A}_2 + \frac{8}{3} \mathrm{A}_3 ;$$

$$\mathrm{M}_0 = \frac{\mathrm{W}\, l}{8} = f_b h' \left(\frac{7 \mathrm{A}_2}{30} + \mathrm{A}_3 \right) ; \therefore \mathrm{W} = \frac{f_b h'}{l} \left(\frac{28}{15} \mathrm{A}_2 + 8 \mathrm{A}_3 \right).$$

In order to obtain a definite result, let $\mathrm{A}_3 = \mathrm{A}_2$; which proportion is not unusual in practice. Then $\mathrm{A} = 4\, \mathrm{A}_2$; and

$$\mathrm{W} = \frac{f_b h' \mathrm{A}}{l} \left(\frac{7}{15} + 2 \right) = \frac{37}{15} \cdot \frac{f_b h' \mathrm{A}}{l}.$$

As before, let $\mathrm{B} = \frac{5}{18} l\, \mathrm{A}$; $f_b = 60{,}000$; $\frac{h'}{l} = \frac{1}{15}$; then

$$\left.\begin{aligned} \frac{s_2\, \mathrm{B}}{\mathrm{W}} &= \frac{l \text{ in inches}}{11{,}840}; \text{ and} \\ .840 \text{ inches} &= 987 \text{ feet nearly}; \end{aligned}\right\} \ldots\ldots (20.)$$

also,

$$11{,}840 : \frac{l}{3} : \frac{11{,}840 - l}{6} : : \mathrm{W} : \mathrm{B} : \mathrm{W}' \ldots\ldots\ldots\ldots (21.)$$

168. **Distribution of Shearing Stress in Beams.** (*A. M.*, 309.)—It has already been shown in Article 160, Division II., how to find the greatest amount of the shearing action of the load at a given cross-section of a beam. Let F denote that amount, A the area of the cross-section at which it acts; then

$$\textit{Mean intensity} \text{ of shearing stress} = \frac{\mathrm{F}}{\mathrm{A}} \ldots\ldots\ldots\ldots (1.)$$

The *distribution* of that stress over the cross-section is such that *its intensity is greatest at the neutral axis*, and gradually diminishes towards the upper and lower surfaces of the beam, where it vanishes. That *greatest intensity* is found by the following process:—Conceive, as in fig. 136, Article 162, p. 250, the cross-section to be divided into thin horizontal layers, such as C; let z be the breadth of any layer, $d\,y$ its depth, y its distance from the neutral axis; also let z_0 be the breadth of the cross section at the neutral axis; I the "*moment of inertia*" of the cross-section, as defined in Article 162, p. 252; y_1 the distance from the neutral axis to *either the upper or the under* surface of the beam; q_0 the required greatest intensity of the shearing stress. Then

$$q_0 = \frac{\mathrm{F}}{\mathrm{I}\, z_0} \int_0^{y_1} y\, z\, d\, y \ldots\ldots\ldots\ldots\ldots\ldots\ldots (2.)$$

The symbol $\int_0^{y_1}$ denotes that the integration or summation of the products $y\, z\, d\, y$ of the area of each layer into its distance from the neutral axis, is to extend from the neutral axis to either the upper or the lower surface of the beam, that integration being thus performed for one only of the two parts into which the neutral axis divides the cross-section. It is a matter of convenience only which of those parts is chosen, as the same result is arrived at in either case.

The maximum intensity of the shearing stress at the given cross-section exceeds the mean intensity in the following proportion:—

$$\frac{q_0\, \mathrm{A}}{\mathrm{F}} = \frac{\mathrm{A}}{\mathrm{I}\, z_0} \int_0^{y_1} y\, z\, d\, y ; \ldots\ldots\ldots\ldots\ldots\ldots\ldots (3.)$$

a ratio depending solely on the figure of the cross-section.

The following table gives some of its values:—

FIGURE OF CROSS-SECTION.	$\frac{q_0 A}{F}$.
I. Rectangle, $z_0 = b$, ...	$\frac{3}{2}$
II. Ellipse and Circle, ...	$\frac{4}{3}$
III. Hollow Rectangle— $A = bh - b'h'$; $z_0 = b - b'$. This includes I-shaped sections,	$\frac{3}{2} \cdot \frac{(bh - b'h') \cdot (bh^2 - b'h'^2)}{(b - b') \cdot (bh^3 - b'h'^3)}$.
IV. Hollow square, $h^2 - h'^2$,..................	$\frac{3}{2}\left(1 + \frac{hh'}{h^2 + h'^2}\right)$.

V. VI. Hollow ellipse and hollow circle; the numerical factor $\frac{4}{3}$; the symbolical factor, the same as for the hollow rectangle and hollow square respectively.

VII. Single T-shaped section; flange A_1; web A_2; $A_1 + A_2 = A$, ... $\frac{3}{2} \cdot \frac{4AA_1 + A_2^2}{A_2(4A_1 + A_2)}$.

VIII. Double T-shaped section; flanges A_1, A_3; web A_2; $A_1 + A_2 + A_3 = A$,

$$\frac{A(24A_1A_3 + 12A_1A_2 + 12A_2A_3) + 3A_2^3 - 12A_1A_2A_3}{A_2(24A_1A_3 + 8A_1A_2 + 8A_2A_3 + 2A_2^2)}$$

When A_1 and A_3, in Case VIII., are large compared with A_2,—that is to say, when a beam consists of strong upper and lower flanges or horizontal bars, connected by a thin vertical web, the shearing force may be treated as if it were entirely borne by the vertical web, and uniformly distributed.

The smallest cross-section of a beam is generally fixed by reasons of convenience, independent of the shearing force to which it is exposed, and is generally much greater than is necessary in order to bear that force. But when it is practicable to adapt the least cross-section of the beam accurately to the shearing force, the preceding formulæ and table furnish the means of doing so, by making

$$q_0 = \frac{f'}{s}; \quad \text{..............................(4.)}$$

where f' is the modulus of rupture by shearing, and s a factor of safety. This equation gives for the least sectional area,

$$A = \frac{q_0 A}{F} \cdot \frac{F}{q_0} = \frac{q_0 A}{F} \cdot \frac{s F}{f'}; \quad \ldots \ldots \ldots \ldots \ldots \ldots (5.)$$

in which formula, $q_0 A \div F$ is to be found by means of equation 3, or of the preceding table of examples.

169. **Deflection of Beams.** (*A. M.*, 300 to 304.)—By the term "*Deflection*," when not otherwise specified, is understood the *greatest displacement* of any point of a loaded beam from its position when the beam is unloaded. Three cases may be distinguished,—that of *Deflection under any load*, that of *Proof Deflection*, or deflection under the greatest load which does not impair the strength of the beam, and that of *Ultimate Deflection*, or deflection immediately before breaking. When the load does not exceed the proof load, the deflection of a given beam, under a load distributed in a given manner, is very nearly proportional to the load: when the proof load is exceeded, the deflection increases in general faster than the load, and in an irregular manner, so that the ultimate deflection is not capable of exact computation. The remainder of this Article will therefore relate to deflections under loads not exceeding the proof load.

The determination of the deflection of a beam under a transverse load is a process which consists of three steps, by which are found successively, the *curvature* at any cross-section, the *slope* at any cross-section, and the *deflection*.

STEP I.—To find the *curvature* at a given cross-section,—divide the bending moment at that cross-section (as found in Article 160, Division III., p. 242) by the "moment of inertia" of that section (as found in Article 162, p. 252), and by the modulus of direct elasticity of the material. The result is the curvature,—that is, the reciprocal of the radius of curvature of a longitudinal line in the beam, which was straight when the beam was unloaded. Denote that radius by r, and the other quantities by the symbols already employed; then

$$\frac{1}{r} = \frac{M}{E I} \quad \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots (1.)$$

The positive or negative sign of this expression will show whether the curvature is concave upwards or downwards.

When the beam is under its proof load, and the given cross-section is that of greatest stress, let M_0 denote the bending moment of that section, and I_0 the moment of inertia; then, as has been shown in Article 162, equations 2 A, 3, 4, and 5, we have

$$M_0 = \frac{f I_0}{y_1}$$

(where f' is the *modulus of proof strength*, or, for most materials, from one-half to one-third of the modulus of rupture, see Article 143, p. 222); so that, in the case now considered, equation 1 becomes

$$\frac{1}{r_0} = \frac{M_0}{E I_0} = \frac{f'}{E y_1} = \frac{f'}{E m' h} \text{ ....(2.)}$$

(m' having the meaning explained in Article 162, p. 252).

This formula gives the *sharpest curvature* which the beam can bear without injury; and as $f' \div E$ is the *proof strain* of the material, that curvature depends on the proof strain, the depth h, and the form of section only.

When the dimensions are all given in inches, the bending moment in inch-pounds, and the moduli of proof strength and elasticity in pounds on the square inch, the radius of curvature will be computed in inches.

The denominator E I in equation 1, expresses the *transverse stiffness or resistance of the beam to bending* at any given cross-section; and as I may be expressed in the form $n' b h^3$ (Article 162, equation 3, p. 252), the resistances of similar cross sections of beams of the same material are to each other as their breadths, and as the cubes of their depths; and consequently,—

The curvatures of beams of the same material at sections of similar figures, under equal bending moments, are inversely as their breadths, and inversely as the cubes of their depths.

Equation 2 also shows that,—

The curvatures of beams of the same material, at sections of similar figures, under their respective proof bending moments, are inversely as their depths simply.

In the case of a *cross section of equal strength* (such as those described in Article 164), equation 2 may be put in the following form:—let f_a' and f_b' be the moduli of proof resistance to cross-breaking by compression and by tearing respectively; then

$$\frac{1}{r_0} = \frac{f_a' + f_b'}{E h} \text{ ....(2 A.)}$$

The curved form assumed by an originally straight longitudinal line in a beam might be drawn approximately by the aid of equation 1, were it not that the great lengths of the radii of curvature, and the smallness of the ordinates of the curve, in all cases which occur in practice, render it neither practicable nor useful to draw the figure of that curve in its natural proportions. But the following process, invented, so far as I am aware, by Mr. C. H. Wild, enables a diagram to be drawn, which represents, with a near

approach to accuracy, that curve, ***with its vertical dimensions exaggerated***, so as to show conspicuously the slopes and ordinates.

Compute, by equation 1, the radii of curvature for a series of equi-distant points in the beam. Diminish all those radii in any proportion which may be convenient, and draw a curve composed of small circular arcs with the diminished radii. Then in the same ratio that the radii, as compared with the horizontal scale of the drawing, are diminished, will the vertical scale of the drawing, according to which the ordinates are shown, be exaggerated.

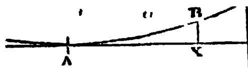


Fig. 146.

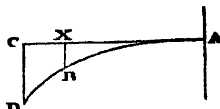


Fig. 147.

STEP II.—To find the *slope*, or inclination of an originally straight longitudinal line in a beam to its original position. The solution of this problem depends on the principle, that the ***difference of slope at two points in that line, is the product of the distance between those points into the mean curvature of the portion of the line between them.*** That is to say, in symbols, let $d\,x$ denote the length of a portion of the line, $1 \div r$ its mean curvature, $d\,i$ the difference of slope at the two ends of that portion; then

$$d\,i = \frac{d\,x}{r} \quad \text{......(3.)}$$

Let i_0 be the slope of the beam at the point taken as the origin of co-ordinates; i, the slope at a point whose distance from that origin is x'; conceive the distance x' to be divided into an indefinite number of small parts, the length of each being $d\,x$; compute by equation 1 the curvature of each of those parts, and by equation 3 the successive differences of slope; sum or integrate those results, and the final result will be the whole difference between the slopes at the origin and at the point x'; that is to say,

$$i = i_0 + \int_0^{x'} \frac{d\,x}{r} \quad \text{......(4.)}$$

When the beam is supported and loaded in such a way that it is known to have *no slope* at a certain point, that point should be taken as the origin. This occurs in two cases; that of a beam fixed at one end and loaded on the projecting portion (fig, 147), which has no slope at the fixed end A; and that of a beam supported at both ends and symmetrically loaded (half shown in fig

146), which has no slope at the middle point A. In these cases, let the tangent A X C at the point of no slope be taken as the axis of abscissæ, along which x' is to be measured; then

$$i_0 = 0; \text{ and } i = \int_0^{x'} \frac{dx}{r} \text{............ ...(5.)}$$

These are the most common cases in practice. In other cases, the slope i_0 at the origin must remain indeterminate until the third step of the solution is performed.

The following principles are the consequences of equation 3, when applied to *similar beams of the same material, under loads similarly distributed:*—

The slopes at corresponding points are as the lengths and curvatures; and therefore,

Under equal loads, *the slopes at corresponding points are directly as the lengths, and inversely as the breadths and cubes of the depths;*

Under the proof loads, *the slopes at corresponding points are directly as the lengths, and inversely as the depths.*

The following formulæ express these principles, as applied to the finding of the *steepest slope* in a given beam, which is in general at the point most distant from the point of no slope; for example, at D, in figs. 146 and 147.

Under a *given load* W;

$$\text{steepest slope } i_1 = \frac{m''' \text{W} c^2}{\text{E} n' b h^3}; \text{...............(6.)}$$

Under the *proof load*,

$$\text{steepest slope } i_1 = \frac{m'' f' c}{\text{E} m' h}; \text{......... ......(7.)}$$

And in sections of equal strength,

$$i_1 = \frac{m'' (f'_a + f'_b) c}{\text{E} h} \text{.... ....(7 A.)}$$

For beams fixed at one end, $c = l$; for beams supported at both ends, $c = l \div 2$.

m' and n' are the factors already explained in Article 162, equations 3 and 4, p. 252, and of which values have been given in the table, p. 254.

m'' and m''' are factors depending on the distribution of the load, the mode of support, and the longitudinal section of the beam. Examples of their values will be given in a table further on. They bear the following relations to each other:—

$$\left.\begin{array}{l}\text{in beams fixed at one end } m''' = m\, m''; \\ \text{in beams supported at both ends } m''' = 2\, m\, m'';\end{array}\right\} \text{.. .(8.)}$$

m being the factor already explained in Article 161, equation 1, p. 247, and of which values have been given in the tables of pp. 245 and 246.

STEP III.—To find the *Deflection*. By this term is to be understood the *depression of the lowest point below the highest point of an originally straight horizontal longitudinal line in the beam.*

Let $d\,x$ be the distance between two points in that line, i the mean slope of the line between them, and $d\,v$ their difference of level; then

$$d\,v = i\,d\,x \text{..........................} (9.)$$

Assume any convenient point in the line in question as the origin of co-ordinates; let x' be the distance of another point from it; conceive that distance to be divided into an indefinite number of small parts, the length of each being $d\,x$; compute, by the second step of the process, the slope of each of those divisions, and by equation 9, the successive differences of elevation of their ends; the sum or integral of those results will be the elevation or depression of the point x' relatively to the origin, according as it is positive or negative; that is to say,

$$v = \int_0^{x'} i\,d\,x \text{..........................} (10.)$$

This equation finally determines the figure assumed by an originally straight longitudinal line in the beam.

In the two cases represented by figs. 146 and 147—that is, when the beam is symmetrically loaded, or fixed at one end—the most convenient point for the origin is still the point of no slope A, and the *deflection* sought is the difference of elevation between that point and the furthest point D, whose distance from it is, in a symmetrically loaded beam, the *half-span*, $l \div 2$, and in a beam fixed at one end, the length of the projecting part, l. Hence, denoting the deflection by v_1,

$$\left.\begin{array}{l} \text{In a symmetrically loaded beam, } v_1 = \int_0^{\frac{l}{2}} i\,d\,x; \\ \text{in a beam fixed at one end, } v_1 = \int_0^{l} i\,d\,x. \end{array}\right\} (11.)$$

In other cases, the most convenient point for the origin of co-ordinates is in general one of the points of support; the fixity of the other point of support, for which $v = 0$, will give an equation from which i_0 in equation 4 may be found, and the positions of the most elevated and depressed points are to be found by the condition that for them the slope $i = 0$. Examples of such problems will be given in the sequel.

The following principles are the consequences of equation 9, when applied to *similar beams of the same material, under loads similarly distributed*:—

Under equal loads, *the deflections are directly as the cubes of the lengths, and inversely as the breadths and cubes of the depths.*

Under the proof loads, *the deflections are directly as the squares of the lengths, and inversely as the depths.*

The following formulæ express these principles:—

Deflection under a given load W,

$$\frac{n''' W c^3}{E n' b h^3}; \quad \ldots\ldots (12.)$$

Proof Deflection,

$$v_1 = \frac{n'' f' c^2}{E m' h} \quad (13.)$$

And in sections of equal strength,

$$v_1 = \frac{n'' (f'_a + f_i) c^2}{E h} \quad \ldots\ldots\ldots\ldots\ldots (13\ \text{A.})$$

For beams fixed at one end, $c = l$; for beams supported at both ends, $c = l \div 2$.

m' and n' are the factors already explained in Article 162, equations 3 and 4, p. 252, and of which values have been given in the table, p. 254.

n'' and n''' are factors depending on the distribution of the load, the mode of support, and the longitudinal section of the beam. They bear the following relations to each other:—

$$\left.\begin{array}{l}\text{in beams fixed at one end, } n''' = m\, n''; \\ \text{in beams supported at both ends, } n''' = 2\, m\, n''\end{array}\right\} \ldots (14.)$$

The following tables give examples of the values of the factors in equations 6, 7, 7 A, 12, 13, and 13 A:—

Case.	Proof Load. Factors for Slope. m''	Proof Load. Factors for Deflection. n''	Any Load. Factors for Slope. m'''	Any Load. Factors for Deflection n'''
A. Uniform Cross-Section.				
I. Constant Moment of Flexure,	1	$\frac{1}{2}$		
II. Fixed at one end, loaded at other,	$\frac{1}{2}$	$\frac{1}{3}$	$\frac{1}{2}$	$\frac{1}{3}$
III. Fixed at one end, uniformly loaded,	$\frac{1}{3}$	$\frac{1}{4}$	$\frac{1}{6}$	$\frac{1}{8}$
IV. Supported at both ends, loaded in middle,	$\frac{1}{2}$	$\frac{1}{3}$	$\frac{1}{4}$	$\frac{1}{6}$
V. Supported at both ends, uniformly loaded,	$\frac{2}{3}$	$\frac{5}{12}$	$\frac{1}{6}$	$\frac{5}{48}$
B. Uniform Strength and Uniform Depth. (See Article 165, pp. 259, 260.) (The curvature of these is uniform).				
VI. Fixed at one end, loaded at other,	1	$\frac{1}{2}$	1	$\frac{1}{2}$
VII. Fixed at one end, uniformly loaded,	1	$\frac{1}{2}$	$\frac{1}{2}$	$\frac{1}{4}$
VIII. Supported at both ends, loaded in middle,	1	$\frac{1}{2}$	$\frac{1}{2}$	$\frac{1}{4}$
IX. Supported at both ends, uniformly loaded,	1	$\frac{1}{2}$	$\frac{1}{4}$	$\frac{1}{8}$
C. Uniform Strength and Uniform Breadth. (See Article 165, pp. 259, 260.)				
X. Fixed at one end, loaded at other,	2	$\frac{2}{3}$	2	$\frac{2}{3}$
XI. Fixed at one end, uniformly loaded,	infinite	1	infinite	$\frac{1}{2}$
XII. Supported at both ends, loaded in middle,	2	$\frac{2}{3}$	1	$\frac{1}{3}$
XIII. Supported at both ends, uniformly loaded,	1·5708	0·5708	0·3927	0·1427

The values of m'' and n'' for beams of uniform strength, as given in the above table, are greater than those which occur in practice, because, in computing the table, no account has been taken of the additional material which is placed at the ends of such beams, in order to give sufficient resistance to shearing (see p. 267).

The error thus arising applies chiefly to m'', the factor for the maximum slope. For the factor for the deflection, n'', the error is inconsiderable, as experiment has shown.

170. The **Proportion of the Greatest Depth of a Beam to the Span** (*A. M.*, 302,) is so regulated, that the proportion of the greatest deflection to the span shall not exceed a limit which experience has shown to be consistent with convenience. That proportion, from various examples, appears to be—

$$\text{For the working load, } \frac{v_1}{l} = \text{from } \frac{1}{600} \text{ to } \frac{1}{1{,}500}.$$

$$\text{For the proof load, } \ldots \frac{v_1}{l} = \text{from } \frac{1}{200} \text{ to } \frac{1}{600}.$$

The determination of the proportion, $h_0 \div l$, of the greatest depth of the beam to the span, so as to give the required stiffness, is effected by the aid of equation 13 of Article 169, p. 273, from which we obtain

$$\frac{v_1}{l} = \frac{n'' f'' c}{2 \mathrm{E} m' h_0} = \frac{n'' f l}{4 \mathrm{E} m' h_0};$$

consequently the required ratio is given by the equation

$$\frac{h_0}{l} = \frac{n'' f}{4 m' \mathrm{E}} \cdot \frac{l}{v_1}, \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (1.)$$

an expression consisting of three factors: a factor, $n'' \div 4 m'$, depending on the distribution of the load and the figure of the beam; a factor, $l \div v_1$, being the prescribed ratio of the span to the deflection; and a factor, $f \div \mathrm{E}$, being the *proof* strain, or the *working* strain, of the material, as the case may be. When the cross-section is one of equal strength, as in Article 164, equation 1 may be put in the following form:—

$$\frac{h_0}{l} = \frac{n'' (f'_a + f'_b) l}{4 \mathrm{E} v_1}. \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (2.)$$

EXAMPLE I.—Let the beam be under its *working load*, uniformly distributed, on a beam of uniform section, alike above and below. Then $n'' = \frac{5}{12}$, $m' = \frac{1}{2}$. Let $\frac{l}{v_1} = 1{,}000$ be the prescribed ratio of

the span to the working deflection. Let the material be wrought iron, for which $\frac{1}{3,000}$ is a safe value for the working strain $\frac{f}{E}$. Then

$$\frac{h_0}{l} = \frac{5}{24} \cdot \frac{1,000}{3,000} = \frac{5}{72} = \frac{1}{14 \cdot 4};$$

which is very nearly the average proportion of depth to span adopted for wrought iron girders in practice.

EXAMPLE II.—Let the beam still be under its working load, uniformly distributed; let the cross section be of equal strength, and let the longitudinal section be one of *uniform strength and uniform depth*. (See Article 165, Case IV, p. 260.) In this case, $n'' = \frac{1}{2}$. Let $l \div v_1$ be still $= 1,000$. The material being wrought iron, and the factor of safety about 6, let $f'_a = 6,000$: $f'_b = 10,000$; and let $E = 29,000,000$. Then

$$\frac{h_0}{l} = \frac{1}{8} \cdot \frac{16,000 \times 1,000}{29,000,000} = \frac{1}{14 \cdot 5};$$

being nearly the same as in the preceding example.

EXAMPLE III.—As in Example II., let the beam be under its working load, uniformly distributed; let the cross-section be of equal strength, and the longitudinal section of uniform strength and uniform depth. Then $n'' = \frac{1}{2}$. Let the material be cast iron; let the factor of safety be 6, and let $f'_a = 13,333$, $f'_b = 3,333$, $E = 16,666,000$. The following are the proportions of greatest depth to length for two different values of the proportion of the greatest deflection to the length:—

$$\text{for } \frac{l}{v_1} = 600, \ \frac{h_0}{l} = \frac{1}{13 \cdot 3}$$

$$\text{for } \frac{l}{v_1} = 800, \ \frac{h_0}{l} = \frac{1}{10}.$$

171. **Summary of the Process of Designing a Beam.**—In designing a beam of a given material, and of a given span, to support a given load, distributed in a given way, the process to be followed may be thus summed up:—

I. Decide to what class of figures the cross-section shall belong; for example, whether it is to be rectangular, similar above and below, T-shaped, double T-shaped, of equal strength, and so forth

(see Article 164, pp. 256 to 259); also, of what kind the longitudinal section is to be: as to which, see Article 165, pp. 259 to 261.

II. Determine the greatest depth, by the considerations mentioned in Article 170, p. 275.

III. Find the shearing force and bending moment at a sufficient number of cross-sections, and the greatest shearing force and bending moment, as in Articles 160, 161, pp. 239 to 249.

IV. Multiply the greatest bending moment by a proper factor of safety; which, for a travelling or otherwise moving load, will, in general, be *six*. This gives the breaking moment for rupture by cross-breaking. In like manner, the greatest shearing force, multiplied by a proper factor of safety, gives the ultimate resistance to shearing at the section where the shearing force is greatest.

V. Determine *provisionally* the product of the extreme breadth and square of the depth at the section of greatest bending moment, by dividing that moment (M_0) by the modulus of rupture of the material (f), and by the proper factor (n). (See Article 162, equation 5, p. 252; also the table of the values of n, p. 254.) That is to say, make

$$b' h^2 = \frac{M_0}{n f} \quad \text{.......................... (1)}$$

Divide this by the square of the depth, already found: the result will be the *provisional extreme breadth*, b'.

In some cases, such as those of T-shaped and double T-shaped sections of equal strength, the above process may not be convenient; and then the *provisional sectional areas* of the different parts of the beam are to be deduced from the required moment of resistance M_0, and the already fixed depth h, by the aid of equations 1, 2, 3, 4, or 5, of Article 163, pp. 255, 256, or of the formulæ of Article 164, pp. 256 to 259.

VI. From the extreme depth and the extreme breadth, or sectional area (as the case may be), at the section of greatest bending moment, find all the other required transverse dimensions of the beam.

VII. Thence compute its weight. If this is a small fraction of the external load, the results already obtained are sufficient.

VIII. But if the weight of the beam forms a considerable part of the load, the results already obtained are *provisional* only, and the breadths (and therefore the sectional areas) are to be increased everywhere in the proportion given by Article 167, equation 1, p. 262. The weight of the beam also will be increased in the same proportion.

By means of equation 2 of the same Article, p. 262, the ratio of the weight of the beam to the external load may be found approxi-

mately so soon as the extreme depth, and the diameter of the cross and longitudinal sections have been fixed; and then the breaking load may be found approximately by equation 3 of the same Article, and used in computing the required ultimate resistance to cross-breaking and to shearing; whence the true breadths and areas of the beam may be found at once. But when this method is followed, the exact weight of the beam should afterwards be computed from the dimensions, to test whether the approximate value is sufficiently near the truth.

IX. The method of Article 168, pp. 266 to 268, may, if necessary, be employed to test whether the cross-section at the points of greatest shearing force is sufficient to resist that force.

172. **Suddenly-applied Load—Swiftly-rolling Load.** (*A. M.*, 306.) —A suddenly-applied transverse load, like the suddenly-applied pull of Article 119, p. 227, produces at first double the maximum stress, and double the strain, which the application of a load gradually increasing from nothing to the amount of the given load would produce.

The action of the rolling load to which a railway bridge is subjected is intermediate, in those cases which occur in practice, between that of an absolutely sudden load and a perfectly gradual load. It has been investigated mathematically by Mr. Stokes, and experimentally by Captain Galton, and the results are given in the Report of the Commissioners on the Application of Iron to Railway Structures.

The additional strain arising, whether from the sudden application or swift motion of the load, is sufficiently provided for in practice by the method already so frequently referred to, of making the factor of safety for the travelling part of the load about double of the factor of safety for the fixed part.

173. The **Resilience or Spring of a Beam** (*A. M.*, 305,) is the *work performed* in bending it to the proof deflection;—in other words, the *energy of the greatest shock* which the beam can bear without injury; such energy being expressed by the product of a weight into the height from which it must fall to produce the shock in question. This, if the load is concentrated at or near one point, is the product of half the proof load into the proof deflection; that is to say, let W be the proof load; then the resilience is

$$\frac{W v_1}{2} \dots\dots\dots\dots (1.)$$

If the load is distributed, the length of the beam is to be divided into a number of small elements, and half the proof load on each element multiplied by the distance through which that element is

moved during the proof deflection of the beam. Let u be that distance; then for beams fixed at one end,

$$u = v;$$

and for beams supported at both ends, .(2.)

$$u = v_1 - v.$$

Let $d\,x$ be the length of an element of the beam; w the intensity of the load on it, per unit of length; then the resilience is

$$\frac{1}{2} \int u\,w \cdot d\,x \ldots\ldots\ldots\ldots\ldots\ldots\ldots (3.)$$

The cases in which the determination of resilience is most useful in practice are those in which the load is applied at one point.

Let the beam be fixed at one end and loaded at the other, c being the length of its projecting part. Then

$$Resilience = \frac{W c_1}{2} = \frac{n\,n''}{2\,m'} \cdot \frac{f^2}{E} \cdot c\,b\,h \ldots\ldots\ldots\ldots (4.)$$

This expression consists of three factors, viz.:—

(1.) The volume of the prism circumscribed about the beam, $c\,b\,h$.

(2.) A *Modulus of Resilience*, $\frac{f'^2}{E}$, of the kind already mentioned in Article 149, p. 227.

(3.) A numerical factor, $\frac{n\,n''}{2\,m'}$; in which n and m' (see p. 252) depend on the form of cross-section of the beam, and n'' (see p. 273), on the form of longitudinal section and of plan. The following are values of this compound factor for a *rectangular cross-section*, for which $n = \frac{1}{6}$, $m' = \frac{1}{2}$, and therefore $\frac{n\,n''}{2\,m'} = \frac{n''}{6}$:—

	6
I. Uniform breadth and depth,	$\frac{1}{18}$.
II. Uniform strength, uniform depth,	$\frac{1}{12}$.
III. Uniform strength, uniform breadth,	$\frac{1}{9}$.

If a beam be supported at both ends and loaded in the middle, its length being $l = 2c$, its proof deflection is the same with that of a beam of the same transverse dimensions and of the length c, fixed at one end and loaded at the other; and its proof load is double of that of the latter beam; therefore, its resilience is double of that of the latter beam. Consequently, for rectangular beams of the half span c, supported at both ends and loaded in the middle, we have the following values for the numerical factor of the resilience:—

	$\frac{n''}{6}$
IV. Uniform breadth and depth,	$\frac{1}{9}$.
V. Uniform strength, uniform depth,	$\frac{1}{6}$.
VI. Uniform strength, uniform breadth,	$\frac{2}{9}$.

174. **Effect of Twisting on a Beam.** (*A. M.*, 320 to 325.)—A full account of the theory of resistance to twisting and wrenching would be out of place in the present treatise, as engineering structures are never subjected to any considerable strain of that kind. For the solution of such questions as commonly occur in practice respecting such structures, the following principles are sufficient:—

I. The *Moment of Torsion* or *Twisting Moment* of a load, means the moment of the pair of equal and opposite couples, which, being applied at different points in the length of a bar, tend to twist the intermediate portion, and, if great enough, to break it by wrenching.

II. The *Ultimate Moment of Resistance* of a bar to wrenching ranges from about once and a-half to twice its Moment of Resistance to cross-breaking.

III. Suppose that the resultant load on a beam, W, and the supporting pressures, act in a plane which, instead of coinciding with the middle longitudinal vertical section of the beam, lies to one side of that section, and parallel to it, at the distance L. Then besides the *bending moment* on each cross-section of the beam (M), found as in Article 160, there is a *Twisting Moment* whose value is,

$$T = P_1 L. \quad \text{(1.)}$$

P_1 being the greatest supporting pressure.

In finding the Moment of Resistance (M_1) required to give the beam sufficient strength, the following formula is near enough to the truth for practical purposes:—

$$M_1 = \sqrt{\left\{\frac{M^2}{4} + \frac{T^2}{4}\right\}} + \frac{M}{2}; \ldots\ldots\ldots\ldots\ldots\ldots (2.)$$

and the dimensions of the beam are to be computed as if this quantity, instead of M, were the bending moment of the load.

175. The **Expansion and Contraction** of long beams (*A. M.*, 309), which arise from the changes of atmospheric temperature, are usually provided-for by supporting one end of each beam on rollers of steel or hardened cast iron. The following table shows the proportion in which the length of a bar of certain materials is increased by an elevation of temperature from the melting point of ice (32° Fahr., or 0 Centigrade) to the boiling point of water under the mean atmospheric pressure (212° Fahr., or 100° Centigrade); that is, by an elevation of 180° Fahr., or 100° Centigrade:—

METALS.

Brass,	·00216
Bronze,	·00181
Copper,	·00184
Gold,	·0015
Cast iron,	·00111
Wrought iron and steel,	·00114 to ·00125
Lead,	·0029
Platinum,	·0009
Silver,	·002
Tin,	·002 to ·0025
Zinc,	·00294

EARTHY MATERIALS.

(The expansibilities of stone from the experiments of Mr. Adie.)

Brick, common	·00355
„ fire,	·0005
Cement,	·0014
Glass, average of different kinds,	·0009
Granite,	·0008 to ·0009
Marble,	·00065 to ·0011
Sandstone,	·0009 to ·0012
Slate,	·00104

TIMBER.

(Expansion along the grain, when dry, according to Dr. Joule, *Proceed. Roy. Soc.*, Nov. 5, 1857.)

Baywood,	·000461 to ·000566
Deal,	·000428 to ·000438

Dr. Joule found that moisture diminishes, annuls, and even reverses, the expansibility of timber by heat, and that tension increases it.

176. **Beam Fixed at Both Ends.** (*A. M.*, 307.)—The particular problems respecting beams, which have been solved in the preceding Articles, have all reference to cases in which the beam is either firmly fixed at one end and loaded on the projecting portion, or simply supported at the two ends and loaded between them, and in which, consequently, the determination of the shearing force and bending moment at each point, and of the curvature, slope, and deflection, are simple and direct processes, proceeding step by step from the determination of one quantity to that of another. In this and the following Articles, problems will be considered in which the shearing force and bending moment depend, to a greater or less extent, on the curvature, slope, and deflection; and in which, consequently, the algebraical process of elimination is often required, two or more unknown quantities having to be determined at once by solving an equal number of equations at the same time.

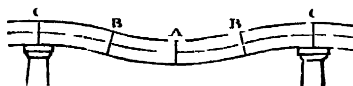


Fig. 148.

A beam is *fixed*, as well as supported at both ends, when a pair of equal and opposite couples are made to act on the vertical sectional planes at its points of support, of magnitude sufficient to maintain its longitudinal axis horizontal there, and so to diminish the deflection, slope, and curvature of its middle portion. This is generally accomplished by making the beam form part of one continuous girder with several points of support, or by making it project on either side beyond its points of support, and so fastening or loading the projecting portions, that their loads, or the resistance of their fastenings, shall give the required pair of couples.

In fig. 148, let C B A B C' represent a beam supported at the points, C, C', loaded along its intervening portion, and so fixed or loaded beyond these points, that at them its longitudinal axis is horizontal, instead of having the slope i'_1, which it would have if the beam were simply *supported* at C', C, and not fixed. At each of the vertical sections above the points of support, C' C, there is an *uniformly-varying horizontal stress*, being a pull above and a thrust below the neutral axis; and the moment of that pair of stresses is that of the pair of equal and opposite couples which maintain the beam horizontal at the points of support. It is required to find,—in the first place, that resisting moment at the vertical planes of support (from which the stress on the material there may at once be found); and secondly, the effect of that moment on the curvature, slope, deflection, and strength of the beam.

The general method of solution of this question is as follows:—Compute, by equation 5 of Article 169, p. 271, i'_1, the slope which the neutral surface of the beam would have at the points, C, C, if it were simply supported there, and not fixed. Then, by the expression $\mathrm{E\,I}\,i'_1 \div c$, find the *uniform* moment of flexure, which if it acted on the beam in such a manner as to make it become convex upwards, would produce a slope at the points, C, C, *equal and contrary* to i_1. This will be the required moment of resistance at the vertical sections C, C. It will afterwards appear that this is the greatest moment of resistance in the beam; so that by putting it instead of M_0 in the formulæ of Articles 162, 163, and 164, pp. 251 to 259, the conditions of strength of the beam are determined. Denote this moment by $-M_1$, the negative sign denoting that it tends to produce convexity upwards, while the load on the beam tends to produce convexity downwards.

Let M be what the moment of flexure at any point of the beam *would be*, if it were simply supported at C, C. Then the actual moment of flexure is

$$\mathrm{M} - \mathrm{M}_1,$$

and by substituting this for M in the equations of Article 169, pp. 268 to 273, the curvature, slope, and deflection, with the proof load, or with any load, are found.

Where M is the greater, as at A, the beam is convex downwards. Where M is the less, as at C, the beam is convex upwards. There are a pair of points, B, B, at which $\mathrm{M} = \mathrm{M}_1$, so that the moment of flexure, and consequently the curvature, vanish, and the beam is subjected to a shearing force alone; these are called the *points of contrary flexure;* and they divide the middle part of the beam, which is convex downwards, from the two endmost parts, which are convex upwards.

EXAMPLE I.—*Symmetrical load on a beam of uniform section, in general.* By Article 169, equation 6, p. 271, observing that $l = 2\,c$, we have

$$i'_1 = \frac{2\,m''\,m}{n'} \cdot \frac{\mathrm{W}\,c^2}{\mathrm{E}\,b\,h^3};$$

And by the table in the same Article, p. 274, Case I.

$$\mathrm{M}_1 = \frac{\mathrm{E\,I}\,i'_1}{c} = \frac{n'\,\mathrm{E}\,b\,h^3\,i'_1}{c};$$

$$= 2\,m''\,m\,\mathrm{W}\,c = m'' \cdot m\,\mathrm{W}\,l = m'' \cdot \mathrm{M}_0 \text{.........(1.)}$$

M_0 being what the bending moment at A *would have been*, had the beam been simply supported.

The values of m'' are given in Article 169, p. 274.

Let M'_0 be the actual bending moment at A. Then

$$M'_0 = (1 - m'')\, M_0 \ \ldots\ldots\ldots\ldots\ldots\ldots (2.)$$

The greatest moment of flexure must be either at A or C, or at both, if the moments of these sections be equal and opposite. But for beams of uniform section, m'' is never greater than $\frac{1}{2}$; therefore the greatest moment of flexure is at C, or both at C and A, and never at A alone.

The *ultimate strength* or greatest moment of resistance of the beam is expressed by the following formula, obtained by putting M_1 instead of $m\ W\ l$, in equation 6 of Article 162, p. 253:—

$$M_1 = m''\, m\, W\, l = n\, f\, b\, h^2; \ldots\ldots\ldots\ldots\ldots\ldots (3.)$$

W being the breaking load, and f the modulus of rupture.

Hence it appears, that *by fixing the ends of an uniform beam so that they shall be horizontal, its strength is increased in the ratio* $1 : m''$.

The *deflection* is found by subtracting that due to the uniform moment M_1, from that, which the load would produce if the beam were simply supported at C and C.

The result is as follows:—

$$v_1 = \left(\frac{n''}{m''} - \frac{1}{2}\right) \cdot \frac{M_1\, c^2}{E\, I} = \left(n'' - \frac{m''}{2}\right) \cdot \frac{M_0\, c^2}{E\, I} \ldots\ldots (4.)$$

For values of n'' see the table already referred to, p. 274. From the last of those expressions, it appears that by fixing the ends horizontal, an uniform beam is made stiffer *under a given load* in the ratio

$$n'' : \left(n'' - \frac{m''}{2}\right).$$

If in the first expression for the deflection, M_1 be considered to represent the moment of resistance corresponding to the proof or limiting safe stress at the section C, we may make $M_1 \div I = f' \div m'\, h$; so as to obtain the following expression for the *deflection under the proof load:*—

$$v_1 = \left(\frac{n''}{m''} - \frac{1}{2}\right) \frac{f\, c^2}{E\, m'\, h} \ldots\ldots\ldots\ldots\ldots\ldots (5.)$$

being less than the proof deflection of a beam simply supported, as given by equation 13, Article 169, p. 273, in the ratio

$$\left(\frac{n''}{m''}-\frac{1}{2}\right):n''.$$

The points of contrary flexure are to be found in each particular case by solving the equation

$$\mathrm{M}-\mathrm{M}_1=0 \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (6.)$$

Examples II. and III. are particular cases of the general problem in Example I.

EXAMPLE II.—*Uniform section, loaded in the middle*

$$m=\frac{1}{4};\ m''=\frac{1}{2};\ n''=\frac{1}{3};$$

$$\mathrm{M}'_0=\mathrm{M}_1=\frac{1}{2}\,\mathrm{M}_0=\frac{1}{8}\,\mathrm{W}\,l=\frac{1}{4}\,\mathrm{W}\,c=n\,f\,b\,h^2; \quad ..(7.)$$

$$v_1=\frac{1}{6}\cdot\frac{f\,c^2}{\mathrm{E}\,m'\,h},\ \left(\frac{n''}{m'}-\frac{1}{2}\right)\div n''=\frac{1}{2}.$$

The points of contrary flexure are midway between A and C.

EXAMPLE III.—*Uniform section, uniformly loaded.*

$$\mathrm{W}=w\,l=2\,c$$

$$m=\frac{1}{8};\ m'=\frac{2}{3};\ n''=\frac{1}{12};$$

$$\left.\begin{aligned}
\mathrm{M}_1&=\frac{2}{3}\,\mathrm{M}_0=\frac{1}{12}\,\mathrm{W}\,l=\frac{1}{6}\,\mathrm{W}\,c=n\,f\,b\,h^2;\\
\mathrm{M}'_0&=\frac{1}{2}\,\mathrm{M}_1=\frac{1}{3}\,\mathrm{M}_0=\frac{1}{24}\,\mathrm{W}\,l;\\
v_1&=\frac{1}{8}\cdot\frac{f\,c^2}{\mathrm{E}\,m'\,h};\ \left(\frac{n''}{m''}-\frac{1}{2}\right)\div n''=\frac{3}{10}.
\end{aligned}\right\}\ \ldots(8.)$$

The points of contrary flexure are each at the following distance from A, the middle point of the beam:—

$$\frac{c}{\sqrt{3}}=0{\cdot}577\,c=0{\cdot}289\,l;. \qquad ..(9.)$$

EXAMPLE IV.—*Uniform strength, uniform depth, uniform load.*

In this case the uniformity of strength is attained by making the breadth at each point proportional to the moment of flexure, as shown in the plan, fig. 149, preserving, at the points of contrary flexure B, B, a sufficient thickness only to resist the shearing force.

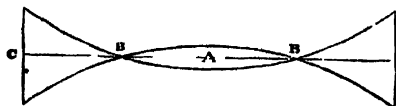


Fig 149.

The curvature of the beam is uniform in amount, changing in direction only at the points of contrary flexure. Therefore, in fig. 148, C B and B A, at each side of the beam, are two arcs of circles of equal radii, horizontal at A and C, and touching each other at B; therefore those arcs are of equal length; therefore each point of contrary flexure B is midway between the middle of the beam A and the point of support C.

It is evident also, that the proof deflection of the beam must be double of that of an uniformly curved beam of half the span, supported at the ends without being fixed; that is to say, one-half of that of an uniformly curved beam of the same span, supported but not fixed; or symbolically

$$v_1 = \frac{1}{4} \cdot \frac{f\, c^2}{E\, m'\, h} \quad \text{.......................(10.)}$$

The actual moment of flexure at A must be the same as in an uniformly loaded beam, with the same intensity of load $w = W \div 2\, c$, supported, but not fixed at B, B; that is to say,

$$M'_0 = \frac{W\, c}{16} = \frac{W\, l}{32} = \frac{M_0}{4} \quad \text{....... ....(11.)}$$

and therefore, the moment of flexure at C is

$$n\, f\, b_1\, h^2 = M_1 = M_0 - M'_0 = \frac{3\, M_0}{4} = \frac{3\, W\, c}{16} = \frac{3\, W\, l}{32}; \quad (12.)$$

b_1 being the breadth of the beam at C, which is three times the breadth b_0 at A.

The breadth b, at any other point, whose distance from A is x, is given by the equation

$$b = \frac{1}{3}\left(1 - \frac{4\, x^2}{c^2}\right) b_1 = \left(1 - \frac{4\, x^2}{c^2}\right) b_0 \quad \text{.........(13.)}$$

In using this equation, the positive or negative sign of the result merely indicates the direction of the curvature.

177. **A Beam Fixed at One End and Supported at Both** is sensibly in the same condition with the part C B A B of the beam in fig. 148.

178. **Continuous Girders.**—The fundamental principle of the theory of continuous girders, with the load distributed in any manner, is the "Theorem of the Three Moments," due originally to Clapeyron and Bresse, and improved by Heppel. (*See* Bresse, *Mécanique Appliquée*, part iii., and the *Proceedings of the Royal Society* for 1869.)

Let ($x = 0$, $v = 0$) and ($x = l$, $v = 0$) be the coordinates of two adjacent points of support of continuous beam, x being horizontal. Let v and the vertical forces be positive downwards.

At a given point x in the span between those points let w be the load per unit of span, and EI the stiffness of the cross-section, each of which functions may be uniform or variable, continuous or discontinuous.

In each of the following double and quadruple definite integrals, let the lower limits be $x = 0$.

$$\left.\begin{array}{l} \iint w dx^2 = m\,; \quad \iint \frac{dx^2}{\mathrm{EI}} = n\,; \\ \iint \frac{x dx^2}{\mathrm{EI}} = q\,; \quad \iint \frac{dx^2}{\mathrm{EI}} \iint w dx^2 = \mathrm{V}. \end{array}\right\} \ldots\ldots\ldots\ldots(1.)$$

When the integrations extend over the whole span l, that will be denoted by affixing 1; for example, m_1, n_1, &c.

Let $-\mathrm{F}$ be the upward shearing force exerted close to the point of support ($x = 0$), M_0 the bending moment, and T the tangent of the inclination, positive downwards, at the same point. Then, by the general theory of deflection, we have, at any point x of the span l, the following equations:—

$$\text{Moment,} \ldots\ldots \mathrm{M} = \mathrm{M}_0 - \mathrm{F}x + m\,; \ldots\ldots\ldots\ldots\ldots\ldots (2.)$$

$$\text{Deflection,} \ldots\ldots v = \mathrm{T}x - \mathrm{F}q + \mathrm{M}_0 n + \mathrm{V}, \ldots\ldots\ldots\ldots (3.)$$

Let M_1 be the moment at the further end of the span l, and suppose it given. This gives the following values for the shearing-force F and slope T at the point ($x = 0$):—

$$\mathrm{F} = \frac{\mathrm{M}_0 - \mathrm{M}_1 + m_1}{l}\,; \ldots\ldots\ldots\ldots\ldots\ldots (4.)$$

and because $v_1 = 0$,

$$\mathrm{T} = \frac{\mathrm{F}q_1 - \mathrm{M}_0 n_1 - \mathrm{V}_1}{l} = \mathrm{M}_0\left(\frac{q_1}{l^2} - \frac{n_1}{l}\right) - \frac{\mathrm{M}_1 q_1}{l^2} + \frac{m_1 q_1}{l^2} - \frac{\mathrm{V}_1}{l} \ldots (5.)$$

Consider, now, an adjacent span extending from the point of support ($x = 0$) to a distance ($-x = l'$) in the opposite direction, and let the definite integrals expressed by the formulæ 1, with their lower limits still at the same point ($x = 0$), be taken for this new span, being distinguished by the suffix -1 instead of 1. Let $-\mathrm{T}'$ be the slope at the point of support ($x = 0$). Then we have for the value of that slope,

$$-\mathrm{T}' = \mathrm{M}_0\left(\frac{q_{-1}}{l'^2} - \frac{n_{-1}}{l'}\right) - \frac{\mathrm{M}_{-1} q_{-1}}{l'^2} + \frac{m_{-1} q_{-1}}{l'^2} - \frac{\mathrm{V}_{-1}}{l'} \ldots\ldots \text{(5A.)}$$

Add together the equations 5 and 5A, and let $t = \mathrm{T} - \mathrm{T}'$ denote the tangent of the small angle made by the neutral layers of the two spans with each other in order to give imperfect continuity. Then, after clearing fractions, we have the following equation, which expresses *the theorem of the three moments*:—

$$\left.\begin{aligned} 0 = \mathrm{M}_0(q_1 l'^2 + q_{-1} l^2 - n_1 l l'^2 - n_{-1} l' l^2) - \mathrm{M}_1 q_1 l'^2 - \mathrm{M}_{-1} q_{-1} l^2 \\ + m_1 q_1 l'^2 + m_{-1} q_{-1} l^2 - \mathrm{V}_1 l l'^2 - \mathrm{V}_{-1} l' l^2 - t l^2 l'^2. \end{aligned}\right\} \ldots \text{(6.)}$$

In a continuous girder of N spans there are $\mathrm{N} - 1$ such equations and $\mathrm{N} - 1$ unknown moments; for the moments at the endmost supports are each $= 0$. The moments at the intermediate points of support are to be found by elimination; which having been done, the remaining quantities required may be computed for any particular span as follows:—The inclination T at a point of support by equation 5; the shearing force F at the same point by equation 4; the deflection v and moment M at any point in that span by equations 3 and 2.

The simplest particular case is that in which the cross-section is uniform, and the piers equidistant. It may be deduced from the general formulæ. (See *Proceedings of the Royal Society*, 1869.) The following, however, is a special demonstration:—

Let fig. 150 represent a viaduct of several spans, consisting of a continuous girder resting at C, C, C, &c., on a series of equidistant

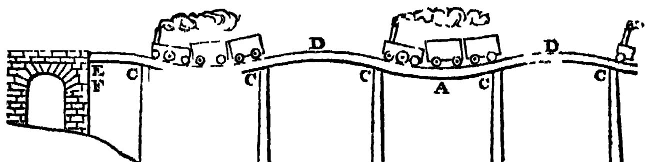


Fig. 150.

piers. The endmost span C E is smaller than the rest; the principle upon which it is to be determined will be afterwards explained.

For the present, the bridge is to be conceived to consist of an indefinitely long series of equal spans, each alternate span only being uniformly loaded from end to end with the greatest possible travelling load.

Let w be the intensity, per lineal foot, of the fixed part of the load; w', that of the travelling part, so that $w + w'$ is the intensity of the load on the more heavily loaded spans, A, A, &c., and w that of the load on the intermediate spans, D, D, &c.

Let $-M_1$ denote the yet unknown negative moment of flexure at the points of support over the piers, C, C, C, &c.

In any heavily loaded division, let horizontal distances denoted by x be measured from the central point A.

In any lightly loaded division, let distances denoted by x' be measured from the central point D.

Let $c = l \div 2$ denote the half span of each bay.

Let the beam be supposed of uniform section, the moment of inertia being I, and the depth h, as before.

Then the following are the results of the processes in Article 169:—

	Lightly loaded Division	Heavily loaded Division.
Bending Moment M =	$\frac{w}{2}(c^2 - x'^2) - M_1$	$\frac{w+w'}{2}(c^2 - x^2) - M_1$
Slope $i = \int \frac{M}{EI} dx =$	$\frac{1}{EI}\left\{\frac{w}{2}\left(c^2 x' - \frac{x'^3}{3}\right) - M_1 x'\right\}$	$\frac{1}{EI}\left\{\frac{w+w'}{2}\left(c^2 x - \frac{x^3}{3}\right) - M_1 x\right\}$

The condition of continuity of the beam above the points of support is, that for $x = c$, and $x' = -c$, the slope i shall be the same. This gives the following equation:—

$$-\frac{w c^3}{3} + M_1 c = \frac{(w + w') c^3}{3} - M_1 c;$$

whence we obtain the following value of the negative moment of flexure above each pier:—

$$M_1 = \frac{(2 w + w') c^2}{6} = \frac{(2 w + w') l^2}{24} \text{ .... (7.)}$$

Introducing this into the expressions for bending moments, and

slope, and proceeding with the processes of Article 169, we obtain the following results:—

	Lightly loaded Division	Heavily loaded Division.
Bending Moment M =	$\frac{w-w'}{6}c^2 - \frac{w}{2}x^2$	$\frac{w+2w'}{6}c^2 - \frac{w+w'}{2}x^2$
Slope i	$\frac{1}{EI}\left\{\frac{w-w'}{6}c^2 x - \frac{w}{6}x^3\right\}$	$\frac{1}{EI}\left\{\frac{w+2w'}{6}c^2 x - \frac{w+w'}{6}x^3\right\}$
Deflection v at any point	$\frac{1}{EI}\left\{\frac{w-2w'}{24}c^4 - \frac{w-w'}{12}c^2x^2 + \frac{w}{24}x^4\right\}$	$\frac{1}{EI}\left\{\frac{w+3w'}{24}c^4 - \frac{w+2w'}{12}c^2x^2 + \frac{w+w'}{24}x^4\right\}$

The following cases of these equations are the most important in practice:—

Bending Moments—

At the centre D of a lightly loaded span,

$$M'_0 = \frac{w-w'}{6}c^2 = \frac{w-w'}{24}l^2;$$

.(8.)

At the centre A of a heavily loaded span,

$$M_0 = \frac{w+2w'}{6}c^2 = \frac{w+2w'}{24}l^2.$$

The *greatest moment of flexure* will be either M_0 at A, or $-M_1$ at C, according as the intensity of the travelling load w', or that of the fixed load w, is the greater.

Central Deflexions under any load—

Of a lightly loaded division, $v'_1 = (w - 2w')\frac{c^4}{24\,EI}$;

(9.)

Of a heavily loaded division, $v_1 = (w + 3w')\frac{c^4}{24\,EI}$.

If w is less than $2w'$, the first of these becomes an *elevation*, being negative.

Central Deflexion of a heavily loaded Division under the Proof Load—

If $w' > w$, so that $M_0 > -M_1$,

$$v_1 = \frac{f'' c^2}{4 E m' h} \cdot \frac{w + 3 w'}{w + 2 w'};$$

if $w > w'$, so that $-M_1 > M_0$; ·(9A)

$$v_1 = \frac{f'' c^2}{4 E m' h} \cdot \frac{w + 3 w'}{2 w + w'}.$$

The corresponding deflections of a lightly loaded division are found by multiplying these expressions by $\frac{w - 2 w'}{w + 3 w'}$.

Points of no curvature occur at the following distances from the centre of each division:—

In a lightly loaded division, at $x' = \pm c \sqrt{\frac{w - w'}{3 w}}$;

In a heavily loaded division, at $x = \pm c \sqrt{\frac{w + 2 w'}{3 (w + w')}}$. (10.)

When those points occur in pairs, they are *points of contrary flexure;* and this is always the case in a heavily loaded span; but in a lightly loaded span, if $w' = w$, there is but one point of no curvature, which is at the middle of the division, and is not a point of contrary flexure; and if $w' > w$, there is no such point in that span.

C E represents a division of the girder, at the end of the viaduct, of such a length that when it is unsupported at E its weight may be at least sufficient to produce the proper moment of flexure $- M_1$ above the nearest pier C. In order that this may be the case, its length C E $= l'$ should be at least sufficient to fulfil the following condition:—

$$w l'^2 = - M_1 = \frac{2 w + w'}{6};$$

and consequently, the least limit of that length is given by the following formula:—

$$\text{C E} = l' \geqq c \sqrt{\frac{2 w + w'}{3 w}} \ldots\ldots\ldots\ldots\ldots (11.)$$

($\geqq$ means "not less than," and $\leqq$ "not greater than.")

The division C E should not extend farther from C than the

farthest point of contrary flexure, when that division has the travelling load on it; that is to say, the greatest limit of its length is

$$C E = l' \leqq c \left(1 + \sqrt{\frac{w + 2w'}{3(w + w')}}\right) \ldots\ldots\ldots\ldots (12.)$$

In order that the fulfilment of these conditions may be possible, the expression (12) must not be less than the expression (11). When the end E of the girder is not supported by the action of the travelling load, it rests on the abutment F.

Throughout the whole of the preceding calculations in this Article, it is to be understood that the *same factor of safety is employed both for the fixed and the travelling parts of the load.* It is considered advisable that the factor of safety for the ordinary working travelling load should be double that for the fixed load (for example, that the former should be 6, and the latter 3). Hence w and w' are to be held to represent the intensities of the two parts of the *proof* load, each being one-third of the corresponding portion of the load which would break the beam if divided in the same manner; so that M_0 or $-M_1$, as the case may be, is one-third of the breaking moment. In the course of the ordinary traffic upon the bridge, the intensity of the fixed load w will continue the same as before, while that of the greatest travelling load will be reduced to one-half of that of the travelling proof load; that is to say, $w' \div 2$.

179. **Rafter, or Sloping Beam with an Abutment.**—In fig. 151, A B represents a straight beam, loaded with weights, and having an abutment at A. The supporting pressures at A and B are to be found by the process explained in Article 112, Case III., p. 174.

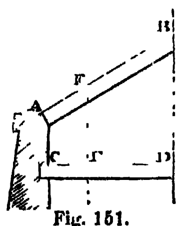


Fig. 151.

Resolve the load and the supporting pressures respectively into components parallel and perpendicular to the beam, or, as they may be called, longitudinal and transverse components. The strain on the beam is compounded of longitudinal compression, produced by the longitudinal forces, and of bending, produced by the transverse forces.

For example, let the load be uniformly distributed, and let w be its intensity in lbs. per lineal inch of the span *measured horizontally;* that is to say, if l denotes the length of the sloping beam between the points of support A, B, i its angle of inclination, and W the total load, the value of that intensity is

$$w = W \div l \cos i \ldots\ldots\ldots\ldots\ldots\ldots\ldots (1.)$$

The supporting pressure at B is horizontal; that at A is inclined at the angle whose tangent is 2 tan i; and their values are respectively—

$$\left.\begin{array}{l}\text{At B, } H = W \div 2 \tan i \\ \text{At A, } \sqrt{H^2 + W^2}. \ldots\end{array}\right\} \ldots\ldots\ldots\ldots\ldots\ldots (2.)$$

The longitudinal components of the load and supporting pressures are as follows:—

$$\left.\begin{array}{l}\text{Of the load, } -W \sin i = -w\,l \cos i \sin i; \\ \text{Of the pressure at B, } -H \cos i = -\dfrac{W \cos^2 i}{2 \sin i}; \\ \text{Of the pressure at A; } H \cos i + W \sin i \\ \qquad = \dfrac{W}{2}\left(\dfrac{1}{\sin i} + \sin i\right),\end{array}\right\} \ldots (3.)$$

the negative signs in the first two expressions denoting downward action. The transverse components are,—

$$\left.\begin{array}{ll}\text{Of the load, } \ldots\ldots\ldots\ldots\ldots & -W \cos i = -w\,l \cos^2 i; \\ \text{Of each of the supporting pressures} \ldots\ldots\ldots\ldots & \dfrac{W \cos i}{2}\end{array}\right\} \ldots (4.)$$

Let A denote the area of a given transverse section of the beam at E, whose distance from B is denoted by x'. Then there is at that section a longitudinal thrust whose intensity, found by dividing its amount by the area A, is as follows:—

$$p' = \left(\frac{W \cos^2 i}{2 \sin i} + w\,x' \cos i \sin i\right) \div A \ldots\ldots\ldots (5.)$$

The bending moment M at the same cross-section is the same as in a beam of the span l, loaded with $w \cos^2 i$ lbs. on the lineal inch *measured along the beam*, and is to be found from these data by the formula of Article 161, Case VI., p. 246, or by the method of Article 176 p. 282, according as the beam is merely fixed in position at A and B, or fixed in *direction* as well as in position.

It may here be remarked that if C D be a horizontal beam of the span $l \cos i$ (being the horizontal projection of the span of A B), loaded with the same uniformly distributed load W as A B, and supported or fixed at the ends in the same manner, the moment of flexure at F, the cross-section corresponding to E, will be the same as at E.

Let I be "the moment of inertia" of the cross-section at E (see

p. 252), and $m'h$ the distance from the neutral axis to the concave side of the beam. Then the moment of flexure M produces an additional thrust at that side of the intensity,

$$p'' = \frac{M m' h}{I} \text{..........................(6.)}$$

so that the greatest intensity of thrust at that cross-section, and the condition that it shall not exceed a safe limit of intensity (f') are expressed as follows:—

$$p' + p'' \leq f' \text{........................(7.)}$$

In the best practical examples, the beam is fixed in direction at A and B; and in that case, the greatest moment of flexure, and the greatest longitudinal thrust, both occur at the abutting joint A. The value of the bending moment being taken from Article 176, equation 8, p. 285, the greatest intensity of thrust is found to be

$$p' + p'' = \frac{W}{2A}\left(\frac{1}{\sin i} + \sin i\right) + \frac{W l \cos i}{12} \cdot \frac{m' h}{I} \text{......(7 A.)}$$

In designing a sloping beam, the depth h may be fixed in the first place, as in Article 170, p. 275. The kind of cross-section adopted will then fix the ratios m', and $I \div m' h^2 A$, I and A themselves being still indeterminate. Let the last of these ratios be denoted by q. Then equation 7 may be put in the following form:—

$$p' + p'' = \frac{W}{A}\left\{\frac{1}{2}\left(\frac{1}{\sin i} + \sin i\right) + \frac{l \cos i}{12 q h}\right\} \text{...... (7 B.)}$$

whence is deduced the following formula for computing the required sectional area:—

$$A = \frac{W}{f'}\left\{\frac{1}{2}\left(\frac{1}{\sin i} + \sin i\right) + \frac{l \cos i}{12 q h}\right\} \text{...........(8.)}$$

Table of Values of $q = I \div m' h^2 A \left(= \frac{nbh}{A}, n \text{ having the values given in Article 163, p. 254}\right)$.

FORM OF CROSS-SECTION.	q.
I. Rectangle, ...	$\frac{1}{6}$
II. Ellipse and Circle, ...	$\frac{1}{8}$

III. Hollow Rectangle, $A = bh - b'h'$; also I-formed section, b' being the sum of the breadths of the lateral hollows,........ $\frac{1}{6}\left(1 - \frac{b'h'^3}{bh^3}\right) \div \left(1 - \frac{b'h'}{bh}\right)$.

IV. Hollow Square, $A = h^2 - h'^2$.. ...... $\frac{1}{6}\left(1 + \frac{h'^2}{h^2}\right)$.

V. Hollow Ellipse,........................ $\frac{1}{8}\left(1 - \frac{b'h'^3}{bh^3}\right) - \left(1 - \frac{b'h'}{bh}\right)$.

VI. Hollow Circle, ... $\frac{1}{8}\left(1 + \frac{h'^2}{h^2}\right)$.

VII. T-formed Section: approximate solution as in Article 163, equation 2, p. 255,—

(Flange A_1; web A_2) .... $\frac{A_2(A_2 + 4A_1)}{6(A_2 + A_1)(A_2 + 2A_1)}$

VIII. Double T-formed section; approximate solution as in Article 163, equation 4, p. 256

(Flanges A_1, A_3; web A_2; the beam supposed to give way by crushing the flange A_1)

$$\frac{A_2(A_2 + 4A_1 + 4A_3) + 12A_1A_3}{6(A_2 + 2A_3)(A_1 + A_2 + A_3)}$$

IX. Double T-formed section, alike above and below ($A_3 = A_1$);..... $\frac{1}{6}\left(1 + \frac{4A_1}{A_2 + 2A_1}\right)$.

When the deflection of the sloping beam A B is compared with that of the horizontal beam C D of equal horizontal span, and under the same load, it appears, from the principle of Article 169, p. 273, that *if those beams are of equal and similar cross section*, their deflections at corresponding points being as the cubes of the lengths, and as the loads producing deflection, which are inversely as the lengths, are to each other as the squares of the lengths; that is

$$\text{Deflection of A B : deflection of C D :: 1 : } \cos^2 i \quad \ldots(9.)$$

Also, the *vertical components* of the deflections are as the lengths simply, or

$$\left.\begin{array}{r}\text{Vertical component of}\\ \text{deflection of A B,}\ldots\end{array}\right\} \text{: deflection of C D :: 1 : } \cos i \ldots(10.)$$

But if A B *be increased in breadth*, as compared with C D in the ratio of 1 : cos i, or sec i : 1, *the vertical components of their deflections will be equal.* This principle will be referred to in the next article.

179 A. To **Deduce the Greatest Stress in a Beam from the Deflection.** —This is done by means of a formula deduced from equation 13 of Article 169, p. 273, as follows:—

Let h be the depth of the beam at the section of greatest stress, and $m' h$ the distance from the neutral axis of that section to that surface of the beam at which the greatest stress is required; m', a factor explained in Article 162, p. 252, depending on the form of cross-section:—

c, the half-span of a beam supported at both ends, or the length of the loaded part of a beam supported at one end;

n'', the factor for proof deflection, explained and exemplified in Article 169, pp. 273, 274;

E, the modulus of elasticity of the material;

v, the observed deflection,

then the intensity of the greatest stress is

$$p_1 = \frac{E\, m' h\, v}{n'' c^2} \quad \text{..........................(1.)}$$

To the values of the factor n'' given in the table, p. 274, may be added the following, which are taken from Articles 176 and 177, pp. 284, 285, 286, 288, and 289.

Cases.	Factors.
XIV. Beam fixed at both ends, section uniform, load in the middle, ..	$\frac{1}{6}$.
XV. Beam fixed at both ends, section uniform, load uniform, ...	$\frac{1}{8}$.
XVI. Beam fixed at both ends, depth uniform, load uniform, strength uniform, ...	$\frac{1}{4}$.
XVII. Beam imperfectly fixed at both ends, section uniform, load uniform, the dead load w being small compared with the rolling load w', and the greatest stress in the middle, ...	$\frac{w + 5w'}{4w + 12w'}$.
XVIII. Beam imperfectly fixed at both ends, section uniform, load uniform, the dead load w being considerable compared with the rolling load w', the lessor of the two following factors (see p. 291),	$\frac{w + 3w'}{4w + 8w'}$; $\frac{w + 3w'}{8w + 4w'}$

180. **Strength and Stiffness of an Arched Rib under Vertical Loads.** —Fig. 152 represents an arched rib, springing from a pair of

abutments, and supposed to be under a vertical load. Let B A C D B′ be a curve traversing the centres of gravity of all the cross-sections of the rib: this may be called the *neutral curve*, and it represents the figure of a "linear arch," or indefinitely thin rib, whose conditions of equilibrium are the same with those of the actual arch. Those conditions have been explained in Article 123, Case II., pp. 186, 187; Article 124, pp. 187, 188; Article 125, pp. 188 to 191; Article 128, pp. 195 to 198; Articles 130, 131, and 132, pp. 199 to 203.

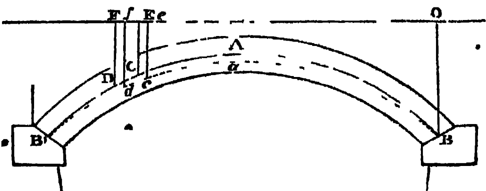


Fig. 152

When a vertical load is distributed over the arch, agreeably to the conditions of equilibrium of the neutral curve, each particle of the arch is compressed, in a direction parallel to a tangent at the nearest point of the neutral curve; and but for the circumstance to be stated presently, that compression would be uniform throughout each cross-section of the rib, so that the neutral curve would be the "line of resistance."

But the compression depresses the whole arch, so that the neutral curve assumes some new figure, such as B a c d B′, in which its curvature at each point differs from the original curvature; and hence, even under a load distributed as for an equilibrated or linear arch, there is a bending action combined with the direct compression. When the distribution of the load differs from that suited to the neutral curve as a linear arch, the bending action varies in its amount and distribution.

In either case the arch acts in the double capacity of a rib under direct compression, and a beam under a transverse load; and its strain and stress at each point are the resultants of the strains and stresses arising from the directly compressive action of the load, and from its bending action.

Problem First. *General Case.*—In solving problems which relate to this subject, it is in general most convenient to measure co-ordinates from a point such as O, in the same vertical line with one end, B, of the neutral curve.

C being any point in the curve, let

$x =$ O E be its horizontal distance from O;

$y =$ E C its vertical depth below O;

Let $l =$ B B′ be the span of the neutral curve, and k its rise.

Let w be the whole intensity of the vertical load, whether constant or variable, in lbs. per inch of horizontal distance, so that

$$\int_0^l w\,d\,x \text{ is the whole load on the arch.}$$

The load $w\,d\,x$ on each small portion of the arch may be conceived to consist of two parts,

$w_1\,d\,x$, producing direct compression alone, being distributed according to the laws of the equilibrium of a linear arch,—that is, in such a manner that $w_1 = \mathrm{H}\,\frac{d^2 y}{d x^2}$ (H being the still undetermined horizontal thrust of the arch), and

$$(w - w_1)\,d\,x = \left(w - \mathrm{H}\,\frac{d^2 y}{d\,x^2}\right) d\,x, \ldots\ldots\ldots\ldots (1.)$$

producing bending.

Having formed the preceding expression, by putting for w and $\frac{d^2 y}{d\,x^2}$ their proper values, proceed as follows:—

The vertical component of the shearing force at any point, such as C, is (see p. 242)—

$$\mathrm{F} = \mathrm{F}_0 - \int_0^x w\,d\,x + \mathrm{H}\left(\frac{d\,y}{d\,x} - \frac{d\,y_0}{d\,x_0}\right) \ldots\ldots\ldots (2.)$$

F_0 being the still undetermined vertical component of the shearing force at B, and $\frac{d\,y_0}{d\,x_0}$ the slope of the neutral curve at that point.

The bending moment at C is (see p. 243)—

$$\left.\begin{array}{c}\mathrm{M} = \mathrm{M}_0 + \int_0^x \mathrm{F}\,d\,x = \mathrm{M}_0 + \mathrm{F}_0\,x - \int_0^x\int_0^x w\,d\,x^2 - \\ \mathrm{H}\left(y_0 - y + x\frac{d\,y_0}{d\,x_0}\right)\end{array}\right\} \ldots (3.)$$

M_0 being the still undetermined bending moment at P

The *alteration of curvature* produced in the neutral curve at C by the bending action is $-\mathrm{M} \div \mathrm{E\,I}$, the negative sign being prefixed to denote that downward curvature is to be considered as positive; and the *alteration of slope* is expressed as follows:—

$$i = \frac{d\,v}{d\,x} = i_0 - \int_0^x \frac{\mathrm{M}}{\mathrm{E\,I}} \sqrt{1 + \frac{d\,y^2}{d\,x^2}} \cdot d\,x; \ldots\ldots\ldots (4.)$$

i_0 being the still undetermined alteration of the slope at B.

The vertical deflection at C is expressed thus,-

$$v = \int_0^x i \, d\, x. \qquad .(5.).$$

The bending action of the load is thus expressed by the four equations, 2, 3, 4, 5, containing four indeterminate constants, H, F_0, M_0, i_0. If, in each of those equations, x be made $= l$, expressions are obtained applicable to the further end of the span, B'. These expressions may be denoted by F_1, M_1, i_1, v_1.

Let $d\,s = \text{C D} = \sqrt{d\,x^2 + d\,y^2}$ denote the length of an indefinitely short arc of the neutral curve. That arc is not altered in length by the bending action of the load; but it is altered by the direct compression in the proportion given by the following equation:—

$$\frac{d\,t}{d\,s} = -\frac{\text{H}\frac{d\,s}{d\,x}}{\text{E A}}; \ldots \ldots \ldots \ldots (6.)$$

in which A denotes the sectional area of the rib at C, and the negative sign indicates compression.

To find the combined effect of the bending action and the compressive action on the figure of the neutral curve, proceed as follows:—

Let u denote the positive horizontal displacement of a point in it, such as C. For example, C D being the original position of an indefinitely short arc, and $c\,d$ its altered position, let

$$\text{O E} = x; \qquad \text{O F} = x + d\,x;$$
$$\text{E C} = y; \qquad \text{F D} = y + d\,y;$$
$$\text{C D} = d\,s;$$
$$\text{O}\,e = x + u; \qquad \text{O}\,f = x + u + d\,x + d\,u,$$
$$e\,c = y + v; \qquad f\,d = y + v + d\,y + d\,v,$$
$$c\,d = d\,s + d\,t.$$

Then from the two equations,

$$d\,s^2 = d\,x^2 + d\,y^2;$$

$$(d\,s + d\,t)^2 = (d\,x + d\,u)^2 + (d\,y + d\,v)^2;$$

The following is deduced:—

$$2\,d\,s \cdot d\,t + d\,t^2 = 2\,d\,x \cdot d\,u + d\,u^2 + 2\,d\,y \cdot d\,v + d\,v^2;$$

and from this the terms $d\,t^2$, $d\,u^2$, $d\,v^2$, may be rejected, as inap-

preciably small compared with the other terms, reducing it to the following:—

$$d\,s \cdot d\,t = d\,x \cdot d\,u + d\,y \cdot d\,v;$$

whence is obtained the following expression for the ***horizontal displacement of* D *relatively to* C**:—

$$d\,u = \frac{d\,s}{d\,x}\,d\,t - \frac{d\,y}{d\,x}\,d\,v \text{ ....(7.)}$$

For $d\,t$ put its value according to equation 6, and make $\frac{d\,s^2}{d\,x^2} = 1 + \frac{d\,y^2}{d\,x^2}$, and $d\,v = i\,d\,x$, then

$$d\,u = -\frac{\mathrm{H}}{\mathrm{E\,A}}\cdot\left(1 + \frac{d\,y^2}{d\,x^2}\right)^{\frac{3}{2}} d\,x - i\,\frac{d\,y}{d\,x}\,d\,x; \text{ ...(7 A.)}$$

which being integrated, gives for the horizontal displacement of C relatively to B and in a direction away from it,

$$\int_0^x \left\{ \frac{\mathrm{H}}{\mathrm{E\,A}}\left(1 + \frac{d\,y^2}{d\,x^2}\right)^{\frac{3}{2}} + i\,\frac{d\,y}{d\,x} \right\} d\,x; \text{.......(8.)}$$

an expression containing the same four indeterminate constants that have already been mentioned; and if x be made $= l$, there is obtained the *alteration of the span* B B′, which may be denoted by u_1.

If the abutments are absolutely immoveable, $u_1 = 0$. If they yield, u_1 may be found by experiment. Hence, as a ***first equation* of condition** for finding the indeterminate constants, we have

$$u_1 = 0, \text{ or a given quantity. ....(9.)}$$

A second equation of condition expresses the immobility in a vertical direction of B′, the further end of the rib, and is as follows:—

$$v_1 = 0. \text{ ....(10.)}$$

The ends of the arched rib are either fixed or not fixed in direction. In the former case, $i_0 = 0$; and in the latter, $\mathrm{M}_0 = 0$; so that in either case, the number of indeterminate constants is reduced to three. One more equation of condition is therefore required; and it is one or other of the following:—

If the ends are fixed in direction, $i_1 = 0$; ....(11.)

if they are not fixed in direction, $\mathrm{M}_1 = 0$. .....(11 A.)

The values of the three constants being found by elimination from the three equations of condition, are to be introduced into the expressions for the moment of flexure (3) and the deflection (5), which will now become formulæ for calculation.

If thrust be treated as positive, and tension as negative, the greatest intensity of stress at any given cross-section is to be computed by the formula,

$$p_1 = \frac{\mathrm{H}\frac{ds}{dx}}{\mathrm{A}} \pm \frac{\mathrm{M}\, m'\, h}{\mathrm{I}}; \quad \text{(12.)}$$

the positive or negative sign being used according as the moment M acts towards or from the edge of the rib under consideration, whose distance from the neutral curve is $m'\,h$.

From the expression 12 may be deduced the position of the point where the stress is greatest for a given arrangement of load, the arrangement of load which makes that stress an absolute maximum, and the corresponding value of the stress.

The *vertical deviation* of the *line of resistance* from the neutral curve at any point is given by the expression

$$\mathrm{M} \div \mathrm{H}; \quad \text{(13.)}$$

and its *perpendicular* or *normal deviation* by the expression

$$\mathrm{M} \div \mathrm{H}\frac{ds}{dx}; \quad \text{(14.)}$$

and these deviations take place in the direction towards which M acts.

When the deflection is found by direct experiment, the following formula may be used to compute the greatest stress from it:—

$$p_1 = \frac{\mathrm{H}\frac{ds}{dx}}{\mathrm{A}} \pm \frac{4\,\mathrm{E}\, m'\, h\, v}{n''\, l^2}; \quad \text{(15.)}$$

the second term being similar to the expression in Article 179 A, p. 296.

The preceding is a general method, applicable to all cases in which the load is vertical. The following particular cases are the most useful in practice:—

PROBLEM SECOND. *Rib of Uniform Stiffness.*—If the depth and figure of the cross-section of an arched rib are uniform, and its breadth is at each point proportional to the secant of the inclination of the rib to the horizon at that point; that is, to

$$\frac{d s}{d x} = \sqrt{1 + \frac{d y^2}{d x^2}};$$

so that if A_1 be the sectional area, and I_1 the moment of inertia, of the rib at the crown, and A and I the corresponding quantities at any other point, we have

$$A = A_1 \sqrt{1 + \frac{d y^2}{d x^2}};\ I = I_1 \sqrt{1 + \frac{d y^2}{d x^2}}; \ldots (16.)$$

then the intensity of the direct thrust along the rib is everywhere equal, and the vertical deflection at each point is the same with that of an uniform straight horizontal beam of the same section with the arched rib at its crown, and acted upon by the same bending moments

This is expressed symbolically by introducing the preceding expressions into equations 4, 6, 8, 12, and 15, which now take the following form :—

$$i = \frac{d v}{d x} = i_0 - \frac{1}{E I_1} \int_0^x M\, d x; \ldots \ldots (4\text{ A.})$$

$$\frac{d t}{d s} = - \frac{H}{E A_1}; \ldots \ldots (6\text{ A.})$$

$$u = - \frac{H}{E A_1} \int_0^x \left(1 + \frac{d y^2}{d x^2}\right) d x - \int_0^x i \frac{d y}{d x} d x; \ (8\text{ A.})$$

In the present case, as well as in all cases in which the depth and figure of section are uniform, it is convenient to express the moment of inertia of the cross-section in terms of its area and depth, as in Article 178, p. 294, by the aid of a factor q, as follows :—

$$I = q\, m'\, h^2\, A; \ldots \ldots (17.)$$

(see the table of values of q, pp. 294, 295); for thus $E A_1$ is rendered a common divisor in the expression (8 A.) for the change of span, which becomes

$$u_1 = \frac{1}{E A_1} \left\{ - H \int_0^l \left(1 + \frac{d y^2}{d x^2}\right) d x + \frac{1}{q\, m'\, h^2} \int_0^l \frac{d y}{d x} \int_0^x M \cdot d x^2 \right\}; \ (18.)$$

while equation 12, for the greatest stress at a given cross-section, becomes

$$p_1 = \frac{1}{A_1}\left\{ H \pm \frac{M}{q\,h} \right\}. \quad \text{............(12 A.)}$$

affording a ready means of computing the requisite area of cross section, when the depth and figure have been fixed beforehand.

Ribs of uniform stiffness are not of common occurrence in practice, but the formulæ relating to them may be applied with little error to flat segmental ribs of uniform section.

PROBLEM THIRD. *Case in which the Abutments yield proportionally to the Horizontal Thrust.*—Let the enlargement of the span of the arch due to horizontal thrust be expressed by the equation

$$u_1 = a\,H\,, \quad \text{........................(9 A.)}$$

then equation 8 takes the following form:—

$$H \left\{ a + \int_0^l \frac{\left(1 + \frac{d\,y^2}{d\,x^2}\right)^{\frac{3}{2}}}{E\,A}\,d\,x \right\} + \int_0^l i\,\frac{d\,y}{d\,x}\,d\,x = 0; \text{(8 C.)}$$

and for a rib of uniform stiffness,

$$H \left\{ a + \frac{1}{E\,A_1}\int_0^l \left(1 + \frac{d\,y^2}{d\,x^2}\right)d\,x \right\} + \int_0^l i\,\frac{d\,y}{d\,x}\,d\,x = 0. \text{ (8 D.)}$$

The co-efficient a may be determined by experiment. For example, in the course of some recent experiments, a stone pier, 24 feet broad and 11 feet thick at the base, was found to yield to the extent of ·27 of an inch to a thrust of 240,000 lbs. applied at a height of 25 feet above its base. In this case, the value of a was

$$\frac{\cdot 27}{240{,}000} = \cdot 000{,}001{,}125.$$

Further experiments are wanting to establish general principles as to the yielding of piers and abutments.

PROBLEM FOURTH. *Parabolic Rib with Rolling Load; the Ends fixed in direction; the Abutments immoveable.*—The following is the most useful case in practice:—Let the neutral curve B A B′ be a parabola, and the rib of uniform depth and uniform stiffness; and let the ends be broad and flat, and accurately bedded on the skewbacks from which they spring, so that their directions may be regarded as fixed; that is to say,

$$i_0 = i_1 = 0. \quad \text{........................(19.)}$$

Take the origin of co-ordinates on a level with the summit of the neutral curve; then the equation of that curve is as follows, k being its rise:—

$$y = \frac{4\,k}{l^2}\left(\frac{l}{2} - x\right)^2; \qquad \text{.................... (20.)}$$

Whence we have—

$$\left.\begin{array}{l} \frac{d\,y}{d\,x} = -\frac{8\,k}{l^2}\left(\frac{l}{2} - x\right); \ \frac{d\,y_0}{d\,x_0} = \frac{-4\,k}{l}; \ 1 + \frac{d\,y^2}{d\,x^2} = 1 + \\ \quad \frac{64\,k^2}{l^4}\left(\frac{l}{2} - x\right)^2; \ \int_0^l \left(1 + \frac{d\,y^2}{d\,x^2}\right) d\,x = l + \frac{16\,k^2}{3\,l}; \\ \qquad \frac{d^2\,y}{d\,x^2} = \frac{8\,k}{l^2}. \end{array}\right\} \text{(21.)}$$

We further find—

$$\int_0^l \frac{d\,v}{d\,x}\frac{d\,y}{d\,x}\,d\,x = \int_0^l \frac{d\,y}{d\,x}\,d\,v = (\text{because } v_0 = v_1 = 0)$$

$$-\int_0^l \frac{d^2\,y}{d\,x^2}\,v\,d\,x = -\frac{8\,k}{l^2}\int_0^l v\,d\,x; \qquad \text{.......... (22.)}$$

being in this case simply proportional to the *area of deflection*, $\int_0^l v\,d\,x$.

Let the rib be under an uniform fixed load, w_0 lbs. on the horizontal lineal inch, and a rolling load of w lbs. on the horizontal lineal inch; the rolling load, covering the horizontal length $r\,l$ of the rib at the end furthest from the origin of co-ordinates, leaves $(1 - r)\,l$ unloaded.

Then equations 2, 3, 4, and 5, become as follows:—formulæ relating to the unloaded division being denoted by A, and those relating to the loaded division by B,—

SHEARING FORCE,—

$$\left.\begin{array}{ll} \text{(A.)} & F_0 + \left(\frac{8\,k\,H}{l^2} - w_0\right) x; \\ \text{(B.)} & F = F_0 + \left(\frac{8\,k\,H}{l^2} - w_0\right) x - w\left\{x - (1 - r)\,l\right\}; \end{array}\right\} \text{(23.)}$$

BENDING MOMENT,—

$$\left.\begin{array}{ll} \text{(A.)} & M = M_0 + F_0\,x + \left(\frac{8\,k\,H}{l^2} - w_0\right)\frac{x^2}{2}; \\ \text{(B.)} & M = M_0 + F_0\,x + \left(\frac{8\,k\,H}{l^2} - w_0\right)\frac{x^2}{2} - \\ & \qquad w\frac{\left\{x - (1 - r)\,l\right\}^2}{2}. \end{array}\right\} \text{......(24.)}$$

ALTERATION OF SLOPE,—

$$\left.\begin{array}{l}\text{(A.)} \quad i = \dfrac{1}{q\, m' h^2\, \mathrm{E\, A}_1} \left\{ -\mathrm{M}_0\, x - \mathrm{F}_0 \dfrac{x^2}{2} - \right. \\ \qquad\qquad \left. \left(\dfrac{8\, k\, \mathrm{H}}{l^2} - w_0 \right) \dfrac{x^3}{6} \right\};\end{array}\right\} (25.)$$

(B.) $\left(\text{to the factor in brackets add} + \dfrac{w}{6} \left\{ x - (1 - r)\, l \right\}^3 \right);$

DEFLECTION,—

$$\left.\begin{array}{l}\text{(A.)} \quad v = \dfrac{1}{q\, m' h^2\, \mathrm{E\, A}_1} \left\{ -\mathrm{M}_0 \dfrac{x^2}{2} - \mathrm{F}_0 \dfrac{x^3}{6} - \right. \\ \qquad\qquad \left. \left(\dfrac{8\, k\, \mathrm{H}}{l^2} - w_0 \right) \dfrac{x^4}{24} \right\}; \\ \text{(B.)} \left(\text{to the factor in brackets add} + \dfrac{w}{24} \left\{ x - (1 - r)\, l \right\}^4 \right),\end{array}\right\} (26.)$$

The equations of condition are the following:—

$i_1 = 0$ gives

$$-\mathrm{M}_0 - \mathrm{F}_0 \frac{l}{2} - \left(\frac{8\, k\, \mathrm{H}}{l^2} - w_0 \right) \frac{l^2}{6} + \frac{w\, r^3\, l^2}{6} = 0, \;.. \quad (27.)$$

$v_1 = 0$ gives

$$-\frac{\mathrm{M}_0}{2} - \mathrm{F}_0 \frac{l}{6} - \left(\frac{8\, k\, \mathrm{H}}{l^2} - w_0 \right) \frac{l^2}{24} + \frac{w\, r^4\, l^2}{24} = 0, \quad (28.)$$

The condition that the abutments are immoveable, or $u_1 = 0$, gives

$$-\left(l + \frac{16\, k^2}{3\, l} \right) \frac{\mathrm{H}}{\mathrm{E\, A}_1} + \frac{8\, k}{l^2} \int_0^l v\, d\, x = 0;$$

and multiplying both sides by $\dfrac{q\, m'\, h^2\, \mathrm{E\, A}_1}{8\, k\, l}$, we have

$$\left.\begin{array}{l} -\dfrac{\mathrm{M}_0}{6} - \dfrac{\mathrm{F}_0\, l}{24} + \dfrac{w_0\, l^2}{120} + \dfrac{w\, r^5\, l^2}{120} - \mathrm{H} \left\{ \dfrac{k}{15} + \dfrac{q\, m'\, h^2}{8\, k} \right. \\ \qquad\qquad \left. \left(1 + \dfrac{16\, k^2}{3\, l^2} \right) \right\} = 0.\end{array}\right\} (29.)$$

By elimination between the three equations of condition, the following results are obtained:—

$$\text{make } \frac{45\, q\, m'\, h^2}{4\, k^2} \cdot \left(1 + \frac{16\, k^2}{3\, l^2} \right) = \mathrm{B}, \;\ldots\ldots\;\ldots \quad (30.)$$

then the horizontal thrust is

$$H = \frac{l^2}{8(1+B)k}\left\{ w_0 + w\,(10\,r^3 - 15\,r^4 + 6\,r^5) \right\} \quad (31.)$$

the bending moment at the unloaded end,

$$-M_0 = \frac{w_0\,l^2}{12} \cdot \frac{B}{1+B} + \frac{w\,l^2}{12}\left\{ 4\,r^3 - 3\,r^4 - \frac{10\,r^3 - 15\,r^4 + 6\,r^5}{1+B} \right\} \quad \ldots (32.)$$

and at the loaded end,

$$-M_1 = \frac{w_0\,l^2}{12} \cdot \frac{B}{1+B} + \frac{w\,l^2}{12}\left\{ 6\,r^2 - 8\,r^3 + 3\,r^4 - \frac{10\,r^3 - 15\,r^4 + 6\,r^5}{1+B} \right\} \quad (33.)$$

The greatest intensity of stress occurs at the loaded end of the rib; and its value is, for thrust;

$$p_1 = \frac{1}{A_1}\left(H + \frac{M_1}{q\,h}\right) = \frac{l^2}{8\,A_1}\left\{ \frac{w_0}{1+B}\left(\frac{1}{k} + \frac{2\,B}{3\,q\,h}\right) + w\left(\frac{2}{3\,q\,h}(6\,r^2 - 8\,r^3 + 3\,r^4) - \frac{10\,r^3 - 15\,r^4 + 6\,r^5}{1+B}\left(\frac{2}{3\,q\,h} - \frac{1}{k}\right)\right)\right\}; \quad (34.)$$

for tension, let p'_1 denote the stress, and q' the value of the factor q; then

$$p'_1 = \frac{1}{A_1}\left(\frac{M_1}{q'\,h} - H\right) = \frac{l^2}{8\,A_1}\left\{ \frac{w_0}{1+B}\left(\frac{2\,B}{3\,q'h} - \frac{1}{k}\right) + w\left(\frac{2}{3\,q'\,h}(6\,r^2 - 8\,r^3 + 3\,r^4) - \frac{10\,r^3 - 15\,r^4 + 6\,r^5}{1+B}\left(\frac{2}{3\,q\,h} + \frac{1}{k}\right)\right)\right\} \quad (35.)$$

Let r_1 denote the value of r which gives the absolute maximum of thrust; r'_1 that which gives the absolute maximum of tension (if any), then

$$= \frac{2}{5} \cdot \frac{1 + B}{1 - \frac{3}{2}\frac{q h}{k}}; \quad r'_1 = \frac{2}{5} \cdot \frac{\quad}{1 + \frac{3}{2}\frac{q' h}{k}}; \qquad ..(36.)$$

and those absolute maxima are,

$$\text{thrust } p_1 = \frac{l^2}{8 A_1} \left\{ \frac{w_0}{1 + B} \left(\frac{2 B}{3 q h} + \frac{1}{k} \right) + \frac{2 w}{3 q h} \left(2 r_1^2 - 2 r_1^3 + \frac{3}{5} r_1^4 \right) \right\} \ldots(37.)$$

$$\text{tension } p'_1 = \frac{l^2}{8 A_1} \left\{ \frac{w_0}{1 + B} \left(\frac{2 B}{3 q' h} - \frac{1}{k} \right) + \frac{2 w}{3 q' h} \left(2 r'^2_1 - 2 r'^3_1 + \frac{3}{5} r'^4_1 \right) \right\} \quad ..(38.)$$

Equation 37 serves to compute the proper sectional area for the rib, when its depth and form have been fixed. If equation 38 gives a negative result, there is no tension at any point of the rib.

The vertical component of the shearing force at the unloaded end is

$$F_0 = \frac{l}{2} \left\{ \frac{w_0 B}{1 + B} + w \left(2 r^3 - \frac{10 r^3 - 15 r^4 + 6 r^5}{1 + B} \right) \right\}; (39.)$$

and this, together with the proper values of $- M_0$ and of H, being substituted in equation 26, enables the deflection at any point to be computed.

When $q h \div k$, $q' h \div k$, and B, are all very small fractions (as is often the case), the following equations are *nearly* true:—

$$r_1 = r'_1 = \frac{2}{5} \qquad .(36\ \text{A.})$$

$$p_1 = \frac{l^2}{8 A_1} \left\{ w_0 \left(\frac{2 B}{3 q h} + \frac{1}{k} \right) + 0{\cdot}138 \frac{w}{q h} \right\}; \ldots(37\ \text{A.})$$

$$p'_1 = \frac{l^2}{8 A_1} \left\{ w_0 \left(\frac{2 B}{3 q' h} - \frac{1}{k} \right) + 0{\cdot}138 \frac{w}{q' h} \right\}. \ldots(38\ \text{A.})$$

When, on the contrary $(1 + B) \div \left(1 - \frac{3}{2}\frac{q h}{k}\right)$ is equal to or greater than $5 \div 2$, the greatest intensity of thrust takes place when the beam is loaded along its whole length; and when

$(1 + B) \div \left(1 + \frac{3\, q'\, h}{2\, k}\right)$ is equal to or greater than $5 \div 2$, the greatest intensity of tension also takes place when the beam is loaded along its whole length; that is to say, $r_1 = r'_1 = 1$; and then we have the following equations:—

$$H = \frac{l^2 (w_0 + w)}{8(1 + B)k}; \quad \ldots\ldots\ldots\ldots\ldots (31\ \text{B.})$$

$$-M_0 = -M_1 = \frac{l^2 (w_0 + w) B}{12 (1 + B)}; \quad \ldots\ldots\ldots\ldots (33\ \text{B.})$$

$$p_1 = \frac{l^2}{8\, A_1} \left\{ \frac{w_0}{1 + B} \left(\frac{2\, B}{3\, q\, h} + \frac{1}{k} \right) + 0{\cdot}192 \frac{w}{q\, h} \right\}; \quad (37\ \text{B.})$$

$$p'_1 = \frac{l^2}{8\, A_1} \left\{ \frac{w_0}{1 + B} \left(\frac{2\, B}{3\, q'\, h} - \frac{1}{k} \right) + 0{\cdot}192 \frac{w}{q'\, h} \right\}. \quad (38\ \text{B.})$$

The effect of an *auxiliary horizontal girder*, made fast to the arched rib at its crown, will be considered further on (pp. 313, 314).

PROBLEM FIFTH. *In the same case, when the Abutments yield to the thrust so as to enlarge the span to the extent* $u_1 = a\, H$; it is only necessary to make, throughout the formulæ of Problem Fourth,

$$B = \frac{45\, q\, m'\, h^2}{4\, k^2} \left(1 + \frac{16\, k^2}{3\, l^2} + \frac{a\, E\, A_1}{l} \right). \quad \ldots\ldots\ldots (40.)$$

PROBLEM SIXTH. *Parabolic Rib of equal stiffness, supported at the ends, but not fixed.*—The formulæ of Problem Fourth are applicable to this case, with the modifications, that M_0 and M_1 are each $= 0$, and that i_0 becomes an indeterminate constant. Hence the following results, in which the terms enclosed in square brackets, [ ], have reference to the *loaded* division of the rib only:—

$$F = F_0 + \left(\frac{8\, k\, H}{l^2} - w_0 \right) x - \left[w \left\{ x - (1 - r) l \right\} \right]; \quad (41.)$$

$$M = F_0\, x + \left(\frac{8\, k\, H}{l^2} - w_0 \right) \frac{x^2}{2} - \left[w \cdot \frac{\left\{ x - (1 - r) l \right\}^2}{2} \right]; \quad \ldots (42.)$$

$$i = i_0 - \frac{1}{q m' h^2 E A_1} \left\{ F_0 \frac{x^2}{2} + \left(\frac{8 k H}{l^2} - w_0 \right) \frac{x^3}{6} - \left[\frac{w}{6} \left\{ x - (1 - r) l \right\}^3 \right] \right\}; \quad (43.)$$

$$v = i_0 x - \frac{1}{q m' h^2 E A_1} \left\{ F_0 \frac{x^3}{6} + \left(\frac{8 k H}{l^2} - w_0 \right) \frac{x^4}{24} - \left[\frac{w}{24} \left\{ x - (1 - r) l \right\}^4 \right] \right\}; \quad (44.)$$

and $v_1 = a H$ denoting the enlargement of the span, as in Problem Fifth, we have,—

$$0 = - \left(1 + \frac{16 k^2}{3 l^2} + \frac{a E A_1}{l} \right) \frac{l H}{E A_1} + \frac{8 k}{l^2} \int_0^l v \, dx; \quad \ldots (45.)$$

which, being multiplied by $q m' h^2 E A_1 \div 8 k$, and proper substitutions made, gives the following equation of condition:—

$$0 = \frac{q m' h^2 E A_1 i_0}{2} - \frac{F_0 l^2}{24} + \frac{w_0 l^3}{120} + \frac{w r^5 l^3}{120} - H l \left\{ \frac{k}{15} + \frac{q m' h^2}{8 k} \left(1 + \frac{16 k^2}{3 l^2} + \frac{a E A_1}{l} \right) \right\}; \quad (46.)$$

The other two equations of condition are as follows:—

$$0 = \frac{q m' h^2 E A_1 v_1}{l} = q m' h^2 E A_1 i_0 - \frac{F_0 l^2}{6} + \frac{w_0 l^3}{24} + \frac{w r^4 l^3}{24} - \frac{H l k}{3}; \quad (47.)$$

$$0 = \frac{M_1}{l} = F_0 - \frac{w_0 l}{2} - \frac{w r^2 l}{2} + \frac{4 k H}{l} \ldots\ldots \quad (48.)$$

Equations 46 and 47 give, by eliminating i_0, and dividing by l^2, the following;—

$$0 = - \frac{F_0}{12} + \frac{w_0 l}{40} + \frac{w l}{120} (5 r^4 - 2 r^5) - \frac{H k}{l} \left\{ \frac{1}{5} - \frac{q m' h^2}{4 k^2} \left(1 + \frac{16 k^2}{3 l^2} + \frac{a E A_1}{l} \right) \right\}; \quad \ldots (49.)$$

and eliminating F_0 between this equation and 48, we obtain the following.—

$$0 = -\frac{w_0 l}{5} - \frac{w l}{10}\left(5r^2 - 5r^4 + 2r^5\right) + \frac{H k}{l}\left\{\frac{8}{5} + \frac{3 q m' h^2}{k^2}\left(1 + \frac{16 k^2}{3 l^2} + \frac{a E A_1}{l}\right)\right\}; \quad \ldots(50.)$$

whence, using the following abbreviation,—

$$C = \frac{15 q m' h^2}{8 k^2}\left(1 + \frac{16 k^2}{3 l^2} + \frac{a E A_1}{l}\right), \quad \ldots\ldots\ldots (51.)$$

we have the following values of the horizontal thrust, and of the other constants,—

$$H = \frac{l^2}{8k(1+C)}\left\{w_0 + \frac{w}{2}\left(5r^2 - 5r^4 + 2r^5\right)\right\}; \ldots(52.)$$

$$F_0 = \frac{l}{2}\left\{\frac{w_0 C}{1+C} + w\left(r^2 - \frac{5r^2 - 5r^4 + 2r^5}{2(1+C)}\right)\right\}; \quad (53.)$$

$$i_0 = \frac{l^3}{24 q m' h^2 E A_1}\left\{\frac{w_0 C}{1+C} + w\left(2r^2 - r^4 - \frac{5r^2 - 5r^4 + 2r^5}{2(1+C)}\right)\right\}. \quad (54.)$$

The *shearing force at the loaded end of the rib* is (with the sign reversed)—

$$P = -F_1 = -F_0 + w_0 l + w r l - \frac{8 k H}{l}$$

$$= \frac{w_0 l}{2} + \frac{w l}{2}\left(2r - r^2\right) - \frac{4 k H}{l} \quad (55.)$$

$$= \frac{l}{2}\left\{\frac{w_0 C}{1+C} + w\left(2r - r^2 - \frac{5r^2 - 5r^4 + 2r^5}{2(1+C)}\right)\right\}.$$

To avoid negative signs in what follows, this is denoted as above by P.

The *greatest bending moment* occurs at a point whose horizontal distance from the loaded end of the rib is

$$l - x = \frac{P}{w_0 + w - \frac{8 k H}{l^2}}; \ldots\ldots\ldots\ldots\ldots(56.)$$

and the value of that greatest bending moment is

$$M' = \frac{P(l - x)}{2} = \frac{P^2}{2(w_0 + w) - \frac{8kH}{l^2}}; \ldots\ldots\ldots (57.)$$

giving, for the greatest stress, a thrust whose intensity is

$$p_1 = \frac{1}{A_1}\left(\frac{M'}{q h} + H\right). \ldots \ldots\ldots\ldots (58.)$$

To find how much of the span of the rib must be loaded, in order to make this stress an absolute maximum, and what that maximum is, the value of r is to be deduced from the equation

$$\frac{d p_1}{d r} = 0 \quad \ldots\ldots \ldots\ldots\ldots (59.)$$

This equation is of the fourteenth order. One of its roots is $r = 1$, which in most cases gives a *minimum* value of p_1. Dividing the equation, therefore, by $1 - r = 0$, it is reduced to the thirteenth order; but it is still too complex to be employed as a formula for practical use.

It appears, however, by trial, that with those proportions which are common in practice, a *close approximation* to the absolute maximum value of the stress p_1 is formed by assuming *one half of the rib to be loaded;* that is—

$$r = \qquad (60.)$$

By introducing this value of r into the preceding formulæ, we obtain the following results:—

$$H = \frac{l^2}{8k(1 + C)}\left(w_0 + \frac{w}{2}\right); \ldots\ldots\ldots (52\text{ A.})$$

$$F_0 = \frac{l}{2}\left(w_0 + \frac{w}{2}\right)\frac{C}{1 + C}; \ldots\ldots\ldots (53\text{ A.})$$

$$i_0 = \frac{l^3}{24 q m' h^2 E A_1}\left\{\left(w_0 + \frac{w}{2}\right)\frac{C}{1 + C} - \frac{w}{16}\right\}; \ldots (54\text{ A.})$$

$$P = \frac{l}{2}\left\{\left(w_0 + \frac{w}{2}\right)\frac{C}{1 + C} + \frac{w}{4}\right\}; \ldots\ldots\ldots (55\text{ A.})$$

$$l - x = \frac{l}{4} \cdot \frac{w + 4\left(w_0 + \frac{w}{2}\right)\frac{C}{1+C}}{w + 2\left(w_0 + \frac{w}{2}\right)\frac{C}{1+C}}; \ldots\ldots\ldots (56\ \text{A.})$$

$$M' = \frac{P(l - x)}{\ } = \frac{l}{64} \cdot \frac{\left\{w + 4\left(w_0 + \frac{w}{2}\right)\frac{C}{1+C}\right\}^2}{+ 2\left(w_0 + \frac{w}{2}\right)\frac{C}{1+C}} . (57\ \text{A.})$$

To illustrate this by a numerical example, let the following data be assumed:—

$k = \frac{1}{8}\, l$; $h = \frac{1}{5}\, k = \frac{1}{40}\, l$; $m' = \frac{1}{2}$;

$q = \frac{1}{3}$ (this value requires an I-shaped section to realize it.

$a = 0$; (that is, let the abutment be immoveable).

Then,

$$C = \frac{1}{80} \times \frac{13}{11} = \cdot 013 \text{ nearly.}$$

Also, let the intensity of the rolling load be equal to that of the lead load, or $w = w_0$. Then

$$H = 1{\cdot}48\ l\ w;$$
$$P = 0{\cdot}13\ l\ w;$$
$$l - x = 0{\cdot}26\ l\ w;$$
$$M' = 0{\cdot}0169\ l^2\ w;$$

(being less than the bending moment due to a load of the intensity w over the whole span, in the ratio of $0{\cdot}135$ to 1).

$$p_1 = \frac{1}{A_1}\left(\frac{M'}{q\,h} + H\right) = \frac{l\,w}{A_1}\left(2{\cdot}03 + 1{\cdot}48\right) = 3{\cdot}54\ \frac{l\,w}{A_1}.$$

Problem Seventh. *To find the greatest Deflection of an Arched Rib*, the greatest value of v is to be taken which corresponds to $i = 0$. It can be deduced from equations 25 and 26 of Problem Fifth, and 43 and 44 of Problem Sixth, that in all ordinary cases to which those problems relate, the absolute maximum deflection occurs in the middle of the rib, when it is loaded over its whole length; that is, when

$$r = 1;\ x = \frac{l}{2}.$$

Then in a rib of uniform stiffness, *fixed in direction at the ends,* we have,

$$H = \frac{l^2 (w + w_0)}{8 k (1 + B)};\ F_0 = \frac{l (w + w_0) B}{2 (1 + B)};$$

$$- M_0 = - M_1 = \frac{l^2 (w + w_0) B}{12 (1 + B)};\ \text{and} \qquad (61.)$$

$$v = \frac{l^4 (w + w_0) B}{384\, q\, m'\, h^2\, E\, A_1 (1 + B)}.$$

In a rib of uniform stiffness, *not fixed* in *direction at the ends,* we have,

$$H = \frac{l^2 (w + w_0)}{8 k (1 + C)};\ F_0 = \frac{l (w + w_0) C}{2 (1 + C)};$$

$$i_0 = \frac{l^3 (w + w_0) C}{24\, q\, m'\, h^2\, E\, A_1 (1 + C)},\ \text{and} \qquad (62.)$$

$$v = \frac{5\, l^4 (w + w_0) C}{384\, q\, m'\, h^2\, E\, A_1 (1 + C)}.$$

In comparing these formulæ with equation 12, of Article 169, p. 273, for the deflection of straight beams under any load, it is to be observed that the total load in the present problem is $l (w + w_0)$, that $l^3 \div 384 = c^3 \div 48$, and that $q\, m'\, h^2\, A_1 = I$. Hence it appears that the deflection of an arched rib of uniform stiffness under an uniformly distributed load, is less than that of a straight beam whose section has the same moment of inertia with that of the arched rib at its crown, in the ratio of

B : 1 + B if the ends are fixed in direction (see pp. 305, 308).

C : 1 + C if the ends are merely supported (see p. 310).

PROBLEM EIGHTH. *Arched Rib of uniform stiffness fixed in direction at the ends, and fixed at the crown to a horizontal beam.*—In fig. 153, let B B′ as before be the arched rib, and E A E′ the horizontal beam. In the spandrils of the arch are vertical struts which transmit the vertical load to the curved rib, and cause the vertical components of the deflection

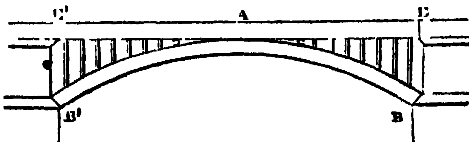


Fig. 153.

of the straight and arched beams to be the same at corresponding points.

The effect of these struts is taken into account by making the total moment of inertia of the cross-section, in the formulæ of Problem Fourth, viz. :—

$$I_1 = q\, m'\, h^2\, A_1,$$

include the moment of inertia of the straight beam · but the area A_1 is still to be that of *the arched beam only*.

Let the curved and straight beams be so firmly connected at the crown (A, fig. 153), that their *horizontal displacement* u is the same at that point; and let the horizontal beam *abut* at its ends, E, E', either against the piers, or against some other part of the superstructure, so as to be capable of resisting a thrust. Then the horizontal thrust is no longer necessarily the same in the two divisions of the arched rib, A B, A B'; but when one of those divisions (as A B') is more heavily loaded than the other, the horizontal thrust in the more loaded division is greater than in the less loaded division, the excess being resisted by that part of the horizontal beam (A E) which is above the less loaded division.

It is unnecessary to give here the complete detailed investigation of this case, or to do more than to state the most important result of that investigation, viz. :—that with the dimensions and under the circumstances that usually occur in practice, the effect of the resistance of the horizontal beam to a longitudinal thrust is to make the greatest intensity of stress in the arched rib under every partial load either less than, or not appreciably greater than, the greatest intensity of stress under a complete load, which thus becomes the absolute maximum of stress in the arched rib, and is given by equations 37 B for thrust, and 38 B for tension, page 308.

The greatest stress in the horizontal beam may be found approximately as follows:—Let h' denote its depth, A' its sectional area,—M_1 the greatest moment of flexure as computed by equation 33 B, p. 308, H the horizontal thrust by equation 31 B, p. 308. Then—

$$\text{greatest thrust, } p'_1 = \frac{M_1 h'}{2\, E I} + \frac{H}{A_1 + A'}; \ldots\ldots\ldots (63.)$$

$$\text{greatest tension, } p'''_1 = \frac{M_1 h'}{2\, E I} \ldots\ldots\ldots\ldots\ldots\ldots (64.)$$

On the subject of the strength of arches in different materials, see the following Articles.—Stone, Article 297, page 432; Timber, Article 345, page 481, and Article 346, page 482; Iron, plain arched ribs, Article 374, page 538; Iron, braced arches, Article 380, page 565.*

* See also Fidler's *Bridge Construction*, 2nd Ed., 1893, and Anglin's *Design of Structures*, 2nd Ed., 1895.

CHAPTER II.

OF EARTHWORK.

SECTION I.—*Strength and Stability of Earthwork in General.*

181. **General Principles — Adhesion — Friction — Natural Slope — Heaviness.**—Earthwork is of two kinds—excavation, or cutting, and filling, or embankment. The term "*earthwork*," in its widest sense, comprehends excavation in rock, as well as in the looser materials of the earth's crust.

Earthwork gives way by the *slipping* or sliding of its parts on each other; and its stability arises from resistance to the tendency so to slip.

In solid rock, that resistance arises from the elastic stress of the material, when subjected to a shearing force; but in a mass of *earth*, as commonly understood, it arises partly from the friction between the grains, and partly from their mutual adhesion; which latter force is considerable in some kinds of earth, such as clay, especially when moist.

But the adhesion of earth is gradually destroyed by the action of air and moisture, and of the changes of the weather, and especially by alternate frost and thaw; so that its friction is the only force which can be relied upon to produce permanent stability.

The temporary additional stability, however, which is produced by adhesion, is useful in the execution of earthwork, by enabling the side of a cutting to stand for a time with a vertical face for a certain depth below its upper edge. That depth is greater the greater the adhesion of the earth as compared with its heaviness; it is increased by a moderate degree of moisture, but diminished by excessive wetness.

The following are some of its values:—

EARTH.	Greatest depth of temporary vertical face.
Clean dry sand and gravel,........................	0
Moist sand, and ordinary surface mould, from	3 to 6 feet.
Clay (ordinary),..............................from	10 to 16 feet.

One of the effects of the temporary stability due to adhesion is

seen in the figure of the surface left after a "slip" has taken place in earthwork. That surface is not an uniform slope, inclined at the angle of repose, but is concave in its vertical section, being vertical at its upper edge, and becoming less and less steep downwards. It is not capable, however, of preserving that figure; for the action of the weather, by gradually destroying the adhesion of the earth, causes the steep upper part of the concave face to crumble down, so that the whole tends to assume an uniform slope in the end.

The *permanent stability* of earth, which is due to friction alone, is sufficient to maintain the side either of an embankment or of a cutting at an uniform slope, whose inclination to the horizon is the *angle of repose*, or angle whose tangent is the *co-efficient of friction.* This is called the *natural slope* of the earth. The customary mode of describing the slope of earthwork is to state the ratio of its horizontal breadth to its vertical height, which is the *reciprocal* of the tangent of the inclination.

Values of the angle of repose (ϕ) and co-efficient of friction (f), and its reciprocal ($1 \div f$), for various substances, have already been given in Article 110, p. 172; but for the sake of convenience, those which refer to the frictional stability of earth are here repeated, with a few additions:—

EARTH		Angle of Repose. ϕ	Co-efficient of Friction f	Customary designation of Natural Slope: $1 \div f$ to 1.
Dry sand, clay, and mixed earth,	from	37°	0·75	1·33 to 1
	to	21°	0·38	2·63 to 1
Damp clay,		45°	1·00	1 to 1
Wet clay,	from	17°	0·31	3·23 to 1
	to	14°	0·25	4 to 1
Shingle and gravel,	from	48°	1·11	0·9 to 1
	to	35°	0·70	1·43 to 1
Peat,	from	45°	1·0	1 to 1
	to	14°	0·25	4 to 1

The most frequent slopes of earthwork are those called 1½ to 1, and 2 to 1; corresponding respectively to the co-efficients of friction 0·67 and 0·5, and to the angles of repose 33⅔° and 26½°, nearly.

The presence of moisture in earth to an extent just sufficient to expel the air from its crevices, seems to increase its co-efficient of friction slightly; but any additional moisture acts like an unguent in diminishing friction, and tends to reduce the earth to a semi-

fluid condition, or to the state of *mud.* In this state, although it has some cohesion, or viscidity, which resists rapid alteration of form, it has no frictional stability; and its co-efficient of friction, and angle of repose, are each of them null.

Hence it is obvious that the frictional stability of earth depends to a great extent on the ease with which the water that it occasionally absorbs can be drained away. The safest materials for earthwork are shivers of rock, shingle, gravel, and clean sharp sand, whether consisting wholly of small hard crystals, or containing a mixture of fragments of shells; for those materials allow water to pass through, without retaining more of it than is beneficial. The cleanest sand, however, may be made completely unstable, and reduced to the state of "quicksand," if it is contained in a basin of water-holding materials, so that water mixed amongst its particles cannot be drained off.

The property of retaining water, and forming a paste with it, belongs specially to clay, and to earths of which clay is an ingredient. Such earths, how hard and firm soever they may be, when first excavated, are gradually softened, and have both their frictional stability and their adhesion diminished by exposure to the air In this respect, mixtures of sand and clay are the worst; for the sand favours the access of water, and the clay prevents its escape.

The properties of earth with respect to adhesion and friction are so variable, that the engineer should never trust to tables or to information obtained from books to guide him in designing earthworks, when he has it in his power to obtain the necessary data either by observation of existing earthworks in the same stratum, or by experiment.

The following are the weights of a cubic foot and of a cubic yard of the ordinary materials of earthwork:—

		Cubic Foot.			Cubic Yard.	
Chalk,....................	from	117 to 174	lbs.	from	3160 to 4730	lbs.
Clay, ...	„	120 to 135	„	„	3240 to 3645	„
Gravel and Shingle,....	„	90 to 110	„	„	2430 to 2970	„
Marl,........................	„	100 to 119	„	„	2700 to 3210	„
Mud,........................		102	„	„	2750	„
Sand, dry,..................		89	„	„	2400	„
„ damp, ...		118	„	„	2190	„
Shale, ...		162	„	„	4370	„

182. **Sides of Rock-Cuttings.**—When rock is firm and sound, so that the permanence of its cohesion may be depended upon, the sides of excavations in it may be made vertical, or nearly so.

How far the cohesion of the rock is to be depended upon, is a question to be solved rather by observation of the rock in each

particular case, than by any general principles having regard to its geological position, mineralogical character, or chemical composition; for the geological position is fixed by the organic remains imbedded in the rock; and these have no connection with its mechanical properties; and rocks composed of the same species of minerals, and the same chemical constituents in the same or nearly the same proportions, show great differences in strength and durability.

It may be observed, however, that the cohesion of igneous and metamorphic rocks, such as granite, syenite, trap, gneiss, mica-slate, marble, quartz-rock, &c., may in general be trusted, unless they are much fissured, or contain potash-felspar, in which cases a sufficient slope must be given, to prevent fragments from falling into the cutting so as to do damage. Of the sedimentary rocks, those which contain much clay, such as shale, are to be treated with caution, how hard soever they may be when first cut; for they are liable to soften by the action of the weather. Sandstone and limestone, whether compact or granular, if fit for building purposes, will stand with vertical or nearly vertical faces; but those materials exist of every degree of hardness, from that of rock, properly speaking, to that of earth. Sandstone is met with which crumbles in the hand, and requires slopes of from 1 to 1 to $1\frac{1}{2}$ to 1; and chalk, according to its degree of hardness and soundness, stands at slopes varying from $\frac{1}{2}$ to 1 to $1\frac{1}{2}$ to 1.

The stability of sedimentary rocks in the side of a cutting is greater when the beds are horizontal, or dip away from the cutting, than when they dip towards it.

183. **Theory of the Stability and Pressure of Loose Earth.** (*A. M.*, 194 to 198.)—The stress exerted in different directions through a given particle in a mass of earth is subject to the general principles which govern the compound internal stress of solids, as already stated in Article 108, pp. 166 to 170.

It is also subject, when friction alone is the cause of stability, to the *limitation* expressed by the following principle:—

I. **General Principle of the Stability of Loose Earth**.—*It is necessary to the stability of a granular mass, that the direction of the pressure between the portions into which it is divided by any plane should not, at any point, make with the normal to that plane an angle greater than the angle of repose.*

The plane in any mass on which the obliquity of the pressure is greatest, is perpendicular to the plane which contains the axes of greatest and least pressure.

Referring to fig. 85, p. 168, and to the description of that figure in pp. 168, 169, it is evident that the above principle is equivalent

to stating that the greatest value of the angle of obliquity $\angle$ N O R or $\hat{nr}$ in that figure shall not exceed ϕ, the angle of repose of the earth in question.

The greatest value of $\hat{nr}$ obviously occurs when O R is perpendicular to P Q, and is given by the following equation:—

$$\max \hat{nr} = \text{arc sin.} \frac{\text{M R}}{\text{O M}} = \text{arc sin} \frac{p_1 - p_2}{p_1 + p_2};$$

and this angle must not exceed the angle of repose; whence the condition of stability of the earth is expressed as follows:—

$$\frac{\text{M R}}{\text{O M}} = \frac{p_1 - p_2}{p_1 + p_2} \angle \sin \phi; \text{..................} (1.)$$

or otherwise as follows:—

$$\frac{p_2}{p_1} \angle \frac{1 - \sin \phi}{1 + \sin \phi}, \text{....................} (1 \text{ A.})$$

which last equation gives the *least* intensity of pressure p_2 in a given direction, that is consistent with the repose of earth through which a pressure of a given intensity p_1 acts at right angles to the first mentioned direction, and serves to determine the least intensity of horizontal pressure which will maintain the stability of a mass of earth through which a vertical pressure of a given intensity acts.

II. **Conjugate Pressures in Earth.**—But it is necessary in some cases to determine the limiting ratio of the intensities of a pair of *conjugate pressures* in a mass of earth, which may or may not be at right angles to each other; and that problem is solved by the following geometrical construction, easily deduced from Proposition IV. of Article 108, p. 168.

In fig. 154, let C represent a section of a prismatic particle of earth, made by the plane of greatest and least pressures. Let that particle be a rhombic prism, on whose faces the pressures are "*conjugate;*" that is to say, let the pressures on the faces which are parallel to D G, act parallel to E F; while the pressures on the faces which are parallel to E F act parallel to D G.

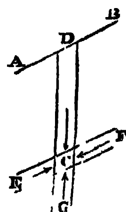


Fig 154.

Let p be the intensity of the pressure parallel to D G, and p' that of the less pressure parallel to E F, each estimated *per unit of area of the plane to which it is conjugate.* Let θ be the angle of obliquity of the prism C;

that is, the difference between each of its angles and a right angle. This angle must not exceed φ, the angle of repose of the earth.

Then the intensities of the conjugate pressures, *per unit of area of planes perpendicular to their directions*, are respectively,—

$$\frac{p}{\cos\theta} \text{ and } \frac{p'}{\cos\theta}.$$

In fig. 155, from one point O, draw two straight lines, O M X and O R, making with each other the angle M O R = φ, the angle of repose. About any convenient point M in one of those straight lines, describe a semicircle Y R X, touching the other straight line in R. (This may be done by describing the dotted semicircle M R O, so as to find the point R.)

Fig. 155.

Through O draw the straight line O Q P, making the angle M O P = θ, the obliquity of the conjugate pressures, and cutting the semicircle Y R X in P and Q. Then the limits of the ratio of the intensities of the conjugate pressures are

$$\frac{\mathrm{O\,Q}}{\mathrm{O\,P}} \text{ and } \frac{\mathrm{O\,P}}{\mathrm{O\,Q}};$$

that is to say, in algebraical symbols,

$$\frac{p'}{p} \text{ cannot be greater than } \frac{\mathrm{O\,P}}{\mathrm{O\,Q}} = \frac{\cos\theta + \sqrt{(\cos^2\theta - \cos^2\varphi)}}{\cos\theta - \sqrt{(\cos^2\theta - \cos^2\varphi)}}, \quad (2.)$$

$$\text{nor less than} \ldots\ldots\ldots\ldots \frac{\mathrm{O\,Q}}{\mathrm{O\,P}} = \frac{\cos\theta - \sqrt{(\cos^2\theta - \cos^2\varphi)}}{\cos\theta + \sqrt{(\cos^2\theta - \cos^2\varphi)}}; \quad (2\text{ A.})$$

being the solution of the problem.

The following are the extreme cases of the problem:—

When the prism C is rectangular, and the conjugate pressures perpendicular to each other, we have $\theta = 0$; O Q P coincides with O Y X, and consequently

$$\frac{p'}{p} \text{ cannot be greater than } \frac{\mathrm{O\,X}}{\mathrm{O\,Y}} = \frac{1 + \sin\varphi}{1 - \sin\varphi}, \ldots\ldots\ldots(3.)$$

$$\text{nor less than} \ldots\ldots\ldots\ldots \frac{\mathrm{O\,Y}}{\mathrm{O\,X}} = \frac{1 - \sin\varphi}{1 + \sin\varphi}. \ldots\ldots(3\text{ A.})$$

When the obliquity of the prism C is the greatest possible, so

that $\theta = \phi$, the points P and Q coalesce in R, and the two limits of the ratio of the conjugate pressures become each equal to unity, giving the single equation,

$$p' = p \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots(4.)$$

III. **Pressure in a Mass of Earth with an unlimited plane upper Surface.**—In fig. 154, p. 319, let A B represent part of the indefinitely extended plane upper surface of a mass of earth, either horizontal, or sloping at any given angle θ not exceeding the angle of repose ϕ. Conceive the whole mass to be divided into layers, such as E F, parallel to A B. The condition of all particles, such as C, into which one of those layers, as E F, can be divided by vertical planes, must be similar; whence it follows that the pressure exerted at any vertical plane is parallel to the surface A B, and the pressure at any surface parallel to A B is vertical. The particle C, formed by the intersection of the vertical column D G with the layer E F, is bounded by conjugate planes; and the conjugate pressures acting through it are respectively vertical, and parallel to the layer.

The vertical pressure p is due to the weight of the column of earth D C which rests on the particle. Let x = D C be its depth, and w the weight of an unit of its volume; then

$$p = w\,x \cos\theta. \qquad (5.)$$

The pressure along the steepest slope of the layer E F, which is exerted through the vertical faces of the prism C, will, if the earth is laid down in layers, be of the *least* intensity sufficient to preserve the repose of the earth, as given by combining equation 2 A with equation 5; that is to say,

$$p' = w\,x \cdot \cos\theta \cdot \frac{\cos\theta - \sqrt{(\cos^2\theta - \cos^2\phi)}}{\cos\theta + \sqrt{(\cos^2\theta - \cos^2\phi)}}. \quad \ldots\ldots\ldots(6.)$$

To represent these results graphically, construct fig. 155 as already described, with O M X horizontal, O R inclined at the "natural slope," and O Q P inclined at the actual slope,—that is, parallel to the steepest slope of the plane A B. From P draw the straight line P W perpendicular to O P, cutting O X in W.

Then

$$\text{O W : O R : O Q :: } w\,x : p : p'. \quad \ldots\ldots\ldots\ldots(7.)$$

The extreme cases are as follows:—

When the upper surface of the earth is horizontal, W and P both coincide with X, and Q with Y; so that,

$$O X : O Y_1 : : w x = p : p'; \text{ and } p' = w x \cdot \frac{1 - \sin \varphi}{1 + \sin \varphi}. \quad (8.)$$

When the upper surface of the earth slopes at the angle of repose, P and Q coincide with R, and W with M; so that

$$O M : O R : : w x : p' = p; \text{ and } p' = p = w x \cos \varphi. \quad (9.)$$

There is a *third conjugate pressure*, exerted horizontally through the particle C, in a direction perpendicular to the vertical plane of steepest slope. Its intensity is represented in fig. 155, by O Y, and is given by the following equation:—

$$p'' = \frac{w x \cdot \cos \theta \, (1 - \sin \varphi)}{\cos \theta + \sqrt{(\cos^2 \theta - \cos^2 \varphi)}}; \text{..............} (10.)$$

and in the two extreme cases it takes the following values:—For a horizontal upper surface, or $\theta = 0$,

$$p'' = p' = w x \frac{1 - \sin \varphi}{1 + \sin \varphi}. \text{ ....} (11.)$$

For the natural slope, or $\theta = \varphi$,

$$p'' = w x \, (1 - \sin \varphi). \text{ ....} (12.)$$

The intensity of the *greatest pressure* exerted through a given particle of earth is represented by O X, and given by the following formula:—

$$p_1 = \frac{w x \cdot \cos \theta \, (1 + \sin \varphi)}{\cos \theta + \sqrt{(\cos^2 \theta - \cos^2 \varphi)}}. \text{...............} (13.)$$

The direction of the *axis of greatest pressure* is at right angles to, and conjugate to, a plane bisecting the angle which a radius drawn from C to Q makes with the horizon; that is to say, the inclination of that axis to the horizon is given by the formula,—

$$\psi = \frac{1}{2} \left(\theta + 180^\circ - \text{arc} \cdot \sin \frac{\sin \theta}{\sin \varphi} \right). \text{ ....} (14.)$$

The extreme cases are,

When the upper surface is horizontal, or $\theta = 0$;

$$p_1 = w x; \; \psi = 90^\circ \text{ (or the axis is vertical). ....} (15.)$$

When $\theta = \phi$;

$$p_1 = w x \, (1 + \sin \phi); \; \psi = \frac{\theta + 90^\circ}{2}; \text{...........} (16.)$$

or the axis of greatest pressure bisects the angle between the slope and the vertical.

The axis of *least pressure* in the plane of greatest slope is perpendicular to that of greatest pressure, and the intensity of the least pressure, being represented by O Y, has already been given in equation 10.

IV. **Pressure of Earth against a vertical Plane.**—In fig. 156, let O X represent a vertical plane in, or in contact with, a mass of earth, whose upper surface Y O Y is either horizontal or inclined at any angle θ, and is cut by the vertical plane in a direction perpendicular to that of steepest declivity. It is required to find the pressure exerted by the earth against that vertical plane *per unit of breadth*, from O down to X, at a depth O X $= x$ beneath the surface, and the direction and position of the resultant of that pressure.

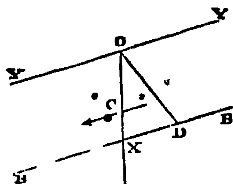


Fig. 156.

The *direction* of that resultant is already known to be parallel to the declivity Y O Y.

Let B B be a plane traversing X, parallel to Y O Y. In that plane take a point D, at such a distance X D from X, that the weight of a prism of earth of the length X D and having an *oblique* base of the area unity in the plane O X, shall represent the intensity of the conjugate pressure per unit of area of a vertical plane at the depth X; that is to say, construct fig. 155 as already described, and make

O P : O Q in fig. 155 : : O X : X D in fig. 156.

Draw the straight line O D; then will the ordinate, parallel to O Y, drawn from O X to O D at any depth, be the length of an oblique prism, whose weight, per unit of area of its oblique base, will be the intensity of the conjugate pressure at that depth. Let O D X be a triangular prism of earth of the thickness unity; the weight of that prism will be the *amount* of the conjugate pressure sought, and a line parallel to O Y, traversing its centre of gravity, and cutting O X in the *centre of pressure* C, will be the *position* of the resultant of that pressure. The depth O C of that centre of pressure beneath the surface is evidently two-thirds of the total depth O X.

To express this symbolically, make

$$\mathrm{X\,D} = x \cdot \frac{p'}{p} = x \cdot \frac{\cos\theta - \sqrt{(\cos^2\theta - \cos^2\phi)}}{\cos\theta + \sqrt{(\cos^2\theta - \cos^2\phi)}}; \quad \text{.......(17.)}$$

then the amount of the conjugate pressure, represented by the weight of the prism O X D, is

$$P' = \frac{w x^2}{2} \cdot \cos \theta \cdot \frac{p'}{p} = \frac{w x^2}{2} \cdot \cos \theta \cdot \frac{\cos \theta - \sqrt{(\cos^2 \theta - \cos^2 \phi)}}{\cos \theta + \sqrt{(\cos^2 \theta - \cos^2 \phi)}}. \quad (18.)$$

In the extreme cases, equations 17 and 18 take the following forms:—For a horizontal surface;

$$\theta = 0; \text{ X D} = x \cdot \frac{1 - \sin \phi}{1 + \sin \phi}; \quad P' = \frac{w x^2}{2} \cdot \frac{1 - \sin \phi}{1 + \sin \phi} \ldots\ldots (19.)$$

For a surface sloping at the angle of repose,

$$\theta = \phi; \text{ X D} = x; \quad P' = \frac{w x^2}{2} \cdot \cos \phi. \ldots\ldots\ldots\ldots (20.)$$

Masses of earth with indefinitely extended plane upper surfaces do not occur in reality; but the formulæ which are applicable to them are applicable to real masses of earth with limited plane upper surfaces, with a degree of accuracy sufficient in most cases for practical purposes. (See *Phil Trans.*, 1856-7).

TABLE OF EXAMPLES.

ϕ	0°	15°	30°	45°	60°
$(90° - \phi) \div 2$	45°	37°½	30°	22°½	15°
$f = \tan \phi$	0	0·268	0·577	1·000	1·732
$1 \div f = \cotan \phi$	∞	3·732	1·732	1·000	0·577
$\sin \phi$	0	0·259	0·500	0·707	0·866
$\frac{1 - \sin \phi}{1 + \sin \phi}$	1	0·588	0·333	0·172	0·072
$\cos \phi$	1	0·966	0·866	0·707	0·500

There is a mathematical theory of the combined action of friction and adhesion in earth; but for want of precise experimental data, its practical utility is doubtful.

SECTION II.—*Mensuration of Earthwork.*

184. **Calculation of Half-breadths and Areas of Land.**—The boundaries of a piece of earthwork in general are as follows:—

I. The *base, forming,* or *formation,* being a surface nearly, and sometimes exactly, horizontal, which forms the bottom of a cutting, or the top of an embankment.

II. The original surface of the ground, which forms the top of a cutting and the bottom of an embankment.

III. The *sides*, or *slopes*, which connect the base with the natural surface, and whose inclination is the steepest consistent with the permanent stability of the material.

Figs. 157, 158, and 159, represent examples of *cross-sections* of

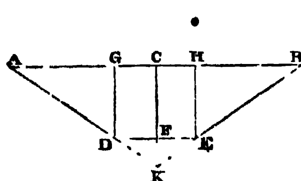


Fig 157.

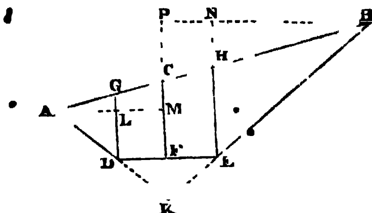


Fig. 158.

pieces of earthwork, in each of which D E is the base, A B the natural surface, and D A and E B are the slopes. In fig. 157, the natural surface is horizontal; in figs. 158 and 159, it slopes sideways, being what is called "side-long ground."

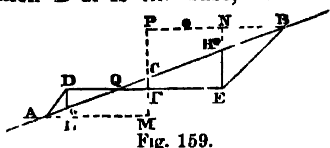


Fig. 159.

Figs. 157 and 158 represent cuttings; to represent embankments, it is only necessary to conceive them to be turned upside down.

Fig. 159 represents a piece of earthwork, of which one side, Q E B, is in cutting called "side cutting," and the other, Q D A, in embankment.

The *half-breadth* of a piece of earthwork has already been mentioned in Article 66, p. 112. It means the horizontal distance from a given point in the *centre line of the base* to one edge of the cutting or embankment; and although it is called "*half*-breadth," it is very generally different at opposite sides of that centre line.

Each half-breadth consists of two parts:—the real half-breadth of the base, which is fixed by the design of the work, and the horizontal breadth of one slope, which is to be found by calculation or by drawing.

In each of the figs. 157, 158, and 159, C represents a point in the centre line, as marked on the ground; F, the point vertically above or below it in the centre line of the base; D G and E H are vertical lines through the edges of the base, D F and F E are the half-breadths of the base.

In fig. 157, where the ground is level across, G A and H B are

the breadths of the slopes, and C A and C B the half-breadths of the earthwork.

In figs. 158 and 159, where the ground slopes sideways, the vertical lines through D, F, and E are produced, if necessary, and are cut at right angles by horizontal lines, A L M, and B N P, drawn through the edges of the earthwork. A L and B N are the breadths of the slopes; and M A and P B are the half-breadths of the earthwork.

When the natural surface of the ground is rugged, the best method of determining the breadths of the slopes of earthwork is by measurement, upon a series of cross-sections of the proposed work, plotted to the same scale horizontally and vertically. (Article 11, p. 11; Article 60, pp. 97, 98.)

When the natural surface of the ground is level, or nearly level across, or has an uniform or nearly uniform sidelong slope, the breadths of slopes may be found by calculation, according to the rules now to be explained.

In each of the following problems, h denotes C F in figs. 157, 158, and 159, being the central depth of the earthwork at the given cross-section; b_0 the *half-breadth* of the base, F D or F E, s to 1, the slope of the earthwork, meaning s horizontal to 1 vertical, b', the half-breadth of the slope.

PROBLEM FIRST. *To calculate the breadth of a slope, when the natural ground is level across.*—In fig. 157,

$$b' = \text{H B} = \text{G A} = s\,h. \quad \ldots\ldots\ldots\ldots\ldots\ldots (1.)$$

PROBLEM SECOND. *To calculate the breadth of a slope when the natural ground has a given uniform sidelong inclination.*

Let the natural sidelong declivity be at the rate of r to 1, that is, let r be the cotangent of the angle which the line A B in figs. 158 and 159 makes with the horizon.

Case I.—When the ground, in proceeding from the centre to the edge of the earthwork, slopes away from the base, as in the right-hand side of figs. 158 and 159—

$$b' = \text{B N} = \frac{r\,s}{r-s}\cdot\left(h+\frac{b_0}{r}\right) \ldots\ldots\ldots\ldots (2.)$$

Here the factor $h+\frac{b_0}{r}$ represents H E, the depth of the earthwork at the edge of the base.

Case II.—When the ground, in proceeding from the centre to the edge of the earthwork, slopes towards the base, as in the left-hand side of fig. 158,—

$$b' = \text{A L} = \frac{r\,s}{r+s} \cdot \left(h - \frac{b_0}{r}\right). \quad \ldots\ldots\ldots\ldots\ldots\ldots(3.)$$

Here the factor $h - \frac{b_0}{r}$ represents G D, the depth of the earthwork at the edge of the base.

Case III.—When the ground intersects the base between the centre line and the edge of the earthwork, as at Q in the left-hand side of fig. 159—

$$b' = \text{A L} = \frac{r\,s}{r-s} \cdot \left(\frac{b_0}{r} - h\right). \quad \ldots\ldots\ldots\ldots\ldots\ldots(4.)$$

Here the factor $\frac{b_0}{r} - h$ represents G D, the depth of the earthwork at the edge of the base.

The horizontal distance of the point Q from the centre line is given by the formula

$$\text{F Q} = r\,h. \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots(5.)$$

It is obvious that the formulæ of this article can be applied to cases in which the slope of the earthwork and the rate of declivity of the ground are different at the two sides of the centre line, as well as to those in which they are the same.

The half-breadths of the earthwork, $b_0 + b'$, to the right and left of the centre line at a given point, being each increased by the breadth required for fencing, give the *total half-breadths* at that point (as stated in Article 66, p. 113); and these being added together, give the *total breadth* of the land to be taken. From a series of those breadths, at different points in the centre line, the *area of land* to be taken may be calculated by the method of ordinates explained in Article 32, pp. 33, 34.

Or the total half-breadths may be plotted on a plan, the boundaries of the land to be taken drawn through them, and the area found by the Method of Triangles, p. 33, or by the use of the Planimeter, p. 34.

185. **Calculation of Sectional Areas of Earthwork.**—The computation of the areas of a series of cross-sections of a piece of earthwork is a step towards calculating its volume, or "quantity." If the ground is rugged, it may be necessary to find the area of each cross-section by measurements made upon a drawing; but if the ground is nearly or exactly level across, or has nearly or exactly an uniform sidelong slope, the area of a given cross-section can be computed from the same data which serve to compute the breadths of the slopes; that is to say,

The natural slope of the ground, ... r to 1;
The slope of the earthwork, ... s to 1;
The half-breadth of the base, ... b_0;
The central depth, ... h.

In each case the area of cross-section required will be denoted by S.

PROBLEM FIRST. *To compute the area of cross-section of a piece of earthwork when the ground is level across*, as in fig. 157.

$$S = FC \cdot GB = h\,(2\,b_0 + b')$$
$$= 2\,b_0\,h + s\,h^2 \text{ ....(1.)}$$

PROBLEM SECOND. *To compute the area of cross-section of a piece of earthwork, when the ground has an uniform sidelong slope, not intersecting the base*, as in fig. 158

The area of the trapezoid G D E H = D E · F C = $2\,b_0\,h$;

" of the triangle B H E = $\frac{BN \cdot HE}{2}$ = (according to Article 184, equation 2) $\frac{r\,s}{2\,(r-s)} \cdot \left(h + \frac{b_0}{r}\right)^2$;

" of the triangle A G D = $\frac{AL \cdot GD}{2}$ = (according to Article 184, equation 3) $\frac{r\,s}{2\,(r+s)} \left(h - \frac{b_0}{r}\right)^2$;

hence, adding those three parts together,

$$S = 2\,b_0\,h + \frac{r\,s}{2\,(r-s)} \cdot \left(h + \frac{b_0}{r}\right)^2 + \frac{r\,s}{2\,(r+s)} \cdot \left(h - \frac{b_0}{r}\right)^2 . \text{ (2.)}$$

This formula may also be put in the following form:—

$$S = \frac{s\,b_0^2 + 2\,r^2\,b_0\,h + r^2\,s\,h^2}{r^2 - s^2} \text{ . ...(3.)}$$

Another mode of expressing the same quantity is as follows,*—

* Suggested, so far as I know, by Mr. Thomas Roberts.

and is convenient for use in connection with a table of squares:— Produce B E and A D till they meet in K, in the vertical line C F produced. Then

$$S = \text{triangle A B K} - \text{triangle E D K}$$
$$= \frac{(AM + BP) \cdot CK - DE \cdot FK}{2};$$

but $FK = \frac{b_0}{s}$; $CK = h + \frac{b_0}{s}$, $DE = 2\, b_0$ and

$$AM + BP = \frac{2\, r^2}{r^2 - s^2}(s\, h + b_0);\ \text{consequently}$$

$$S = \frac{r^2\, s}{r^2 - s^2}\left(h + \frac{b_0}{s}\right)^2 - \frac{b_0^2}{s}. \quad \ldots\ldots\ldots\ldots (4.)$$

PROBLEM THIRD. *To compute the areas of the two divisions of a cross-section of earthwork, when the ground intersects the base*, as in fig. 159.

The cross-section here consists of two similar triangles, Q B E and Q A D, one of which is in cutting and the other in embankment. In the figure, the larger triangle is in cutting; the same figure inverted will represent the case in which the larger triangle is in embankment. When Q, C, and F coincide, the triangles are equal.

Let S′ denote the larger and S″ the smaller triangle. Then

$$S' = \frac{(BP + FQ) \cdot EH}{2} = \frac{(b_0 + r\, h)^2}{2\,(r - s)}; \ldots\ldots (5.)$$

$$S'' = \frac{(AM - FQ) \cdot DG}{2} = \frac{(b_0 - r\, h)^2}{2\,(r - s)}. \ldots\ldots (6.)$$

186. Calculation of Volumes or Quantities of Earthworks.

CASE I. *When two cross-sections* S_0, S_1, *are given, with the longitudinal distance between them* x, the volume (V) of the earthwork between those cross-sections is given approximately, by the following formula, provided S_0 and S_1 are nearly equal, but not otherwise:—

$$V = x \cdot \frac{S_0 + S_1}{2}. \ldots\ldots\ldots\ldots\ldots (1.)$$

CASE II. *When three equidistant cross-sections* S_0, S_1, S_2, *are given, with the total length* x, *of the piece of earthwork between them*, the best approximation is,

$$V = x \cdot \frac{S_0 + 4 S_1 + S_2}{6}. \qquad (2.)$$

CASE III. *Two cross sections given, and one assumed.*—Equation 2 may also be used to give a closer approximation than equation 1, when the two endmost cross-sections only are given, S_0 and S_2, by putting for S_1 the area of an *assumed* cross-section midway between S_0 and S_2; its central depth being assumed to be a mean between the central depths of S_0 and S_2, and the sidelong slope of the ground (if any), at S_1 a harmonic mean between those at S_0 and S_2.

When the ground is level across, this last process gives the following result.—

Let h_0 be the central depth at S_0;

" h_2 " " " at S_2;

then the assumed central depth at S_1 is $\frac{h_0 + h_2}{2}$; and

$$V = x \left\{ b_0 (h_0 + h_2) + s \cdot \frac{h_0^2 + h_0 h_2 + h_2^2}{3} \right\}. \ldots (3.)$$

This formula is called the "Prismoidal Formula." Another form of the same formula, convenient for use in connection with a table of squares, is as follows:—

$$V = x \left\{ b_0 (h_0 + h_2) + s \left\{ \frac{(h_0 + h_2)^2}{4} + \frac{(h_0 - h_2)^2}{12} \right\} \right\}. (4.)$$

Formula 3 is the basis of Sir John Macneill's earthwork tables; formula 4 of Mr. Henderson's.

CASE IV. *An even number of equidistant cross-sections given*, S_0, S_1, S_2, &c. . . . S_m, the distance from section to section being Δx.

$$V = \Delta x \left\{ \frac{S_0}{2} + S_1 + S_2 + \&c. \ldots + \frac{S_m}{2} \right\}. \ . (5.)$$

CASE V. *An odd number of equidistant cross-sections given*, S_0, S_1, S_2, &c. . . . S_n; the distance from section to section being Δx.

$$V = \frac{\Delta x}{3} \left\{ S_0 + 4 S_1 + 2 S_2 + 4 S_3 + 2 S_4 + \&c. \ldots + 2 S_{n-2} + 4 S_{n-1} + S_n \right\}. \quad (6.)$$

Besides the earthwork tables already mentioned, many others have been published, such as Mr. Bidder's, Mr. Haskoll's, &c. Such tables generally give either the *mean sectional area* of a piece of earthwork of a given base and slope, and of given depths at the two ends, or a number proportional to it; which mean area or number being multiplied by the length, gives the volume.

Quantities of earthwork, in Britain, are usually stated in *cubic yards*, while their dimensions are given in feet. The expressions for volumes in this Article, being suited for the case in which the unit of volume is the cube described upon the linear unit, require to be divided by 27, when the dimensions are in feet, to reduce the volumes to cubic yards.

Sometimes, while the breadths and depths are given in feet, the lengths are stated in chains of 66 feet; and in that case, to give the volumes in cubic yards, the expressions in this Article should be multiplied by $\frac{66}{27} = \frac{22}{9} = 2{\cdot}44\dot{4}$.

On the mensuration of earthwork, the treatise of Mr. John Warner may be consulted with advantage.

SECTION III.—*Of the Execution of Earthwork.*

187. **Borings and Trial Shafts.**—The ordinary method of ascertaining the nature of the material to be excavated, previous to the undertaking or execution of any piece of earthwork, is by boring a vertical hole of about $3\frac{1}{2}$ or 4 inches in diameter in the ground, and bringing up specimens of the materials pierced through at different depths.

Inasmuch, however, as the specimens of materials so brought up are, in general, reduced to chips or to powder by the action of the boring-tool, and sometimes to paste or mud by the action of the water which is poured into the hole to keep the tool cool, and facilitate its working in hard strata, the information obtained by boring is not wholly satisfactory; for although it shows the mineralogical composition of the materials found at different depths, it leaves their probable stability in earthwork doubtful, except in so far as it can be inferred from the resistance met with by the boring tool; and this source of information is available to the engineer or contractor at second-hand only, through the statements of the borers. The smallness of the hole, too, makes the results of borings doubtful; for what seems to be a stratum of rock may sometimes prove to be only a solitary block or boulder.

To ascertain completely the nature and qualities of the materials of an intended cutting, trial-shafts or pits should be sunk down to the level of its bottom. The expense and time required for sinking

shafts make it impracticable to use them exclusively. The best method is to combine shafts with borings, by sinking, in every important proposed cutting, one shaft at least, which should in general be at the point of greatest depth, and making, besides, a series of borings at points 200 or 300 yards apart. These borings will be sufficient to show whether any change in the strata occurs sufficient to make it advisable to sink one or more additional shafts in a given cutting.

Boring tools are made of wrought iron, steeled at the cutting edges and points. They are usually about 3 feet long, or a little more, about one half of the length being the tool or boring instrument proper, and the remainder the shank, which is a bar of 1⅛th inch square or thereabouts, having a screw at its upper end to connect it with the first of the *lengthening rods.* These are square bars, usually about 10 feet long, of the same diameter with the shank of the boring tool, with screws at their ends by which they can be united together to any length required by the depth of the bore. The uppermost rod is capable of being hung by a swivel and rope from a triangle or shears set up over the bore-hole, in order to haul up the rods when required. The working part of the tool is made of various figures, for penetrating various materials. The commonest forms are the *auger*, the *worm*, and the *jumper*. The *auger*, which is used for boring all ordinary earths, shale, and soft rock, is formed like a hollow cylinder, about 3½-inches in diameter, with an open sharp edged slit along one side, and slightly contracted at the lower end, which sometimes (for boring soft rock) has a small spiral point like that of a gimlet. It brings up specimens of the material bored in the inside of its hollow cylindrical body.

The *worm* is a sharp pointed spiral, used for boring rock too hard to be pierced by the auger. After the rock has been pierced by the worm, the auger is used to enlarge the hole and bring up the fragments.

Both the auger and the worm are worked by turning them continuously round towards the right (that is, in the direction of the motion of the hands of a watch), by means of a cross-head six feet long, or thereabouts, driven by two men.

To pierce rock that is too hard for the worm, a *jumper* is used. Jumpers are of various figures; some flat, like a chisel, with a sharp edge at the lower end; some square, with a four-sided pyramidal point, like a poker, some spear-pointed. The jumper is worked by raising it a short distance and letting it drop, turning it a little way round after each blow. It is sometimes simply hung by a rope, instead of being screwed to the lower end of the lengthening rods. The materials broken by the jumper are sometimes brought

up by the auger, sometimes by a sort of bucket on the top of the jumper itself.

Bores in very soft materials sometimes require to be lined with a series of cast or wrought iron pipes, pushed down as the bore proceeds, to prevent its sides from falling in; the lowest pipe having a sharp serrated edge. These pipes may be made to screw together, so that they can be hauled up again.

The depth of a layer of moss, mud, or quicksand, at the surface of the ground, is sometimes probed or sounded with a long slender iron rod called a *pricker*.

The operations of sinking shafts will be described further on, under the head of TUNNELLING.

In marking the results of borings and trial shafts on a section (see Article 11, p. 10, and Article 17, p. 15), care is to be taken to show nothing on the paper except the facts actually observed, all conjectural sections of the strata lying between the borings and pits, whether marked by outlines, colour, shadings, or words, being rigidly excluded. The insertion of such conjectural sections, although it improves the appearance of the drawing, and makes it more readily intelligible, is done at the risk of misleading contractors, and involving the companies and engineers in heavy responsibility. The result of the pits and borings being shown exactly as observed, contractors and others are left to draw their own conclusions as to the intermediate strata.

188. **Equalizing Earthwork** is a term applied to the process of so adjusting the formation level of an intended work, that the earth from the cuttings shall be as nearly as possible sufficient to make the embankments, and no more. The art of making this adjustment by the eye upon a section of the ground with sufficient accuracy is soon acquired by practice. In most cases it is essential to economy in the cost of the work; for any surplus of embankment over cutting must be made up from "side cutting;" and the earth from any surplus of cutting over embankment must be formed into "spoil banks;" both of which works involve additional cost for labour and land. But cases sometimes occur, in which it is more economical to make an embankment from side-cutting close at hand, than to bring the necessary material from a far distant cutting on the line of works, or in which it is more economical to throw part of the material from a cutting into a spoil bank, than to send it to a far-distant embankment on the line of works; and these points must be decided by the engineer to the best of his judgment in each particular case.

189. The **Temporary Fencing**, erected before the earthwork is commenced, should enclose all the ground required for the undertaking; that is to say, it should run along the outer boundary of

the strip of land which is to be taken beyond the edge of the earthwork, and whose breadth is added to the half-breadths of the earthwork in calculating and setting out the total half-breadths (p. 113). In the open country, where the permanent fence is to be a hedge and ditch, the breadth of that strip of land is usually about *nine feet;* but where ground is valuable, as amongst gardens and pleasure-grounds, and in towns and suburbs, smaller breadths are used, as to which no general rule can be laid down.

The temporary fence usually consists of posts and rails of larch or oak; the posts being from 4 feet to 4 feet 6 inches apart, about 6 feet long, driven from 2 feet to 2 feet 6 inches into the ground, from 4 to 6 inches broad in a direction *across* the fence, and about 3 inches thick in a direction *along* the fence; the rails about 9 or 10 feet long, 3 inches deep, and 1½ inch thick, scarfed in mortises in every second post. Sometimes the posts in which the rails are scarfed are made stronger than the intermediate posts, and have diagonal stays to increase their stability, the foot of each stay being nailed to a small stake about 2 feet long.

The best site for permanent marks of the line and levels is near the fence (pp. 110, 111).

190. If the soil is wet, a **Catchwater Drain** may be made at the same time with the temporary fencing, at one or both sides of the earthwork, commencing at its outfall into an existing main drain or water course, and working upwards. When the ground has a sidelong slope the catchwater drain is indispensable at the up-hill side of the earthwork. Thus, in fig. 160, A B is part of the base and B C one of the slopes of an intended cutting; C G is part of the natural ground, sloping downwards towards C; D is a catchwater

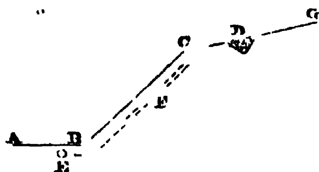


Fig. 160.

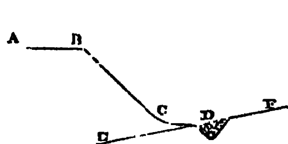


Fig. 161.

drain, to prevent surface water running from G towards C from injuring the slope of the cutting. In fig. 161, A B is part of the base and B C one of the slopes of an embankment; E F is part of the natural ground, sloping downwards from F towards E; D is a catchwater drain, to prevent surface water running from F towards C from collecting at C and injuring the embankment. The catch-water drain may be an open ditch, in ordinary cases from 3 to 4

feet wide and from 2 to 3 feet deep; or it may be an underground drain, built of stone or brick, or made of earthenware tubes (as in the figures), with broken stone or clean gravel above it.

191. **Stripping the Soil.**—The soil or vegetable mould should be stripped from the site of an intended piece of earthwork, and laid down near the fence, in order that it may be afterwards used to re-soil the slopes of the earthwork. The usual depth of soil spread on these slopes varies from 3 to 6 inches.

192. **General Operations of Cutting.**—Where there is no reason to the contrary, it is desirable that the base of a cutting should have a declivity towards the point at which the work of excavation is commenced; for this renders more easy the removal of the earth in wagons, and the temporary and permanent drainage.

A cutting is usually commenced (if the earth will stand for a time with vertical sides) by making a "gullet," or vertical-sided excavation, wide enough to contain one or more lines of temporary rails for the passage of earth wagons. The widening of the cutting to its full width, and the formation of the slope, should be carried on so as never to be far behind the head or most advanced end of the gullet; for the strain thrown on a mass of earth by standing for a time with a vertical face has a tendency to produce cracks, which may extend beyond the position of the intended slopes, and so render the sides of the cutting liable to slip after they have been finished. The advanced end of a cutting of considerable depth, and the parts of its sides whose slopes have not been finished, consist, while the work is in progress, of a series of steps or stages called "lifts," rising one above another by six or eight feet, or thereabouts, the excavators working at the faces of these lifts so as to carry them on together.

From faces at the end or sides of the gullet, the earth is shovelled directly into the wagons. From the other faces of the cutting, the earth is wheeled in barrows along planks to points from which it can be tipped into the wagons.

193. **Draining the Base and Slopes.**—At the foot of each slope of a cutting it is almost always necessary to have a longitudinal drain called a "side-drain," of from 6 inches to 2 feet deep according to the circumstances of the case. It may be a small open ditch, or a channel pitched and faced with stone, or a covered stone or brick drain, or a line of tubes (as at E, fig. 160) with broken stone or gravel above. It may receive the waters of branch drains running across the base, should such be found necessary, and also of branch drains laid in the slopes, as F, fig. 160. When the latter are tubes, they may in general be laid about 2½ feet below the surface. It is in general advisable so to place the side-drain E that its bottom shall not be below the prolongation of the plane

of the slope B C, unless there is a retaining wall; otherwise it may cause that slope to slip, and may itself be crushed or choked.

Springs rising in cuttings require special drains to carry away their waters.

194. The **Labour of Earthwork**, in ordinary cases, consists of *getting*, or excavating; *filling* into barrows or wagons; *wheeling* in barrows; *leading* in wagons; and *teeming* or *tipping*—that is, depositing the earth in the embankment where it is to rest. Other processes required in special cases will be considered farther on.

The labour of *getting* the earth depends mainly upon its adhesion. Loose sand and gravel, soft vegetable mould and peat, can be dug with the shovel or the spade alone; stiffer kinds of earth require to be loosened with the pick before being shovelled into barrows, and in some cases, with crowbars, wedges, or stakes; the softest kinds of rock can be broken up with the pick or crowbar; harder kinds require the action of wedges; harder still, especially if free from natural fissures, need blasting by gunpowder, which will be treated in a separate article.

Wheeling in barrows is performed upon planks, whose steepest inclination should not exceed 1 in 12, unless the men are assisted by means of ropes and winding machinery.

Leading is performed upon light temporary rails, in wagons called "earth-wagons," whose bodies can be tipped over by turning on a pair of horizontal trunnions, so as to empty the earth out: they are drawn by horses or by small locomotive engines.

The labour of shovelling *a given weight* of earth into barrows, and that of wheeling it from the face of the cutting to a given point, tipping it into the wagons, and leading it a given distance, are nearly the same for most ordinary kinds of earth. For *a given bulk* of earth, the labour of those operations varies nearly as the heaviness of the earth.

In order to execute an excavation with speed and economy, it is necessary to fix correctly both the absolute and the proportionate numbers of pickmen, shovellers, and wheelers, or barrowmen, so that all shall be constantly employed. The only method of doing this *exactly*, in any particular case, is by trial on the spot; but an approximation may be made beforehand by estimating from the data of experience.

The *absolute number of excavators* working at the face of a cutting is determined by the horizontal extent of face at which cutting is in progress at once; one excavator to five or six feet of breadth of face, is about as close as they can be placed without getting in each others' way.

The proportion of *wheelers to shovellers* may be estimated ap-

proximately by the fact, that a shoveller takes about as long to fill an ordinary barrow with earth as a wheeler takes to wheel a full barrow about 100 or 120 feet, on a horizontal plank, and return with an empty barrow.

If the full barrow has to be wheeled up an ascent, each foot of rise is to be considered equivalent to six additional feet of horizontal distance.

Hence the following approximate formula:–

Let l be the horizontal distance that the earth has to be wheeled, and h the height of ascent, if any; then

$$\text{number of wheelers to one shoveller} = \frac{l + 6h}{\text{from 100 to 120 feet}} \quad (1.)$$

The number of *barrows* required for each shoveller is one more than the number of wheelers.

A shoveller will throw each shovelful of earth from 6 to 10 feet horizontally, or from 4 to 5 feet vertically upwards. If the earth is to be thrown by the shovel to greater distances or heights, two or more ranks of shovellers must be employed.

The proportion of the pickmen to the shovellers (in a single rank) depends on the stiffness of the earth. The following are examples:—

	Pickmen to one Shoveller.
Loose sand and vegetable mould,...............	0
Compact earth,..............................	$\frac{1}{3}$
Ordinary clay, ...	from $\frac{1}{2}$ to 1
Hard clay,.......................................	„ $1\frac{1}{2}$ to 2.

Earth is designated as "earth of one man," if one shoveller can keep one line of wheelers at work; "earth of a man and a-half," if two shovellers and a pickman are needed to keep two lines of wheelers at work; "earth of two men," if one shoveller and one pickman can keep one line of wheelers at work; and generally, "earth of so many men," according to the number of shovellers and pickmen together who are required to keep one line of wheelers at work. Let m denote that number; then the total number of shovellers, pickmen, and wheelers for each line of wheelers, will be approximately

$$M = m + \frac{l + 6h}{\text{from 100 to 120 feet}} \ \ldots\ldots\ldots\ldots\ldots (2.)$$

The rate at which the cutting may be expected to advance, if no special difficulties occur, may be estimated for each line of wheelers (or for each shoveller in one rank), at about

20 cubic yards of loose sand, or mould, } per day.
or 16 cubic yards of clay, or compact earth, }

The labour of excavating is often considerably lessened, especially in widening a gullet at the sides, by undermining large masses of earth from below, and loosening them by driving stakes behind them from above. This is called "falling."

An earth-wagon holds about as much as 50 wheel-barrows, and if drawn at the walking pace of a horse, its speed may be taken as about one fifth greater than that of the wheel-barrows; so that it is equivalent to about 60 wheel-barrows; and one earth-wagon going and returning a distance of about 6,000 feet horizontally, while another stands to be filled, will keep one shoveller at work. If loaded wagons have to be drawn up an ascent, and the temporary rails are in moderately good order, each foot of ascent may be considered as equivalent to about 150 feet of additional horizontal distance. Hence let L be the horizontal distance in feet to which the earth is to be led in earth-wagons drawn by horses, H the ascent, in feet, if any; then the *number of shovellers* (in single rank) *to each earth-wagon in motion at one time*, is about,

$$\frac{6,000}{L + 150 H} \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots \text{(3.)}$$

and the reciprocal of this expresses the *earth-wagons or fractions of an earth-wagon in motion at one time per shoveller*: but additional wagons, as to which no precise rule can be laid down, must be provided, in order to allow for those which are standing to be filled, and for those which are in the act of being tipped and reversed. With locomotive engines the speed can be increased, and the number of wagons proportionally diminished. The preceding calculations have reference to wagons which hold from 2 to 2¼ cubic yards of earth, or thereabouts, the weight of which is from 2½ to 3 tons, the weight of the wagon itself being between a ton and a ton and a-half. The friction being taken at 15 lbs. per ton (or 1 -- 150th of the gross load nearly), the force required to draw a wagon, or train of wagons, either on a level, or up or down a given declivity, can easily be calculated. In estimating the number of horses required, the force which a horse can exert when walking slowly may be estimated at about 120 lbs.

When the leading of earth is performed, not in wagons on temporary rails, but in two-wheeled one-horse carts on an ordinary road-way, the number of such carts required may be approximately computed, from the data, that the net load of each such cart is about equal to that of twelve wheel-barrows, and its average speed going and returning about one-sixth part more than that of a

wheel-barrow, so that each cart in motion is equivalent to about fourteen wheel-barrows in motion. In this as in the preceding case, in computing the total number of carts, allowance must be made for those which are standing to be filled, and those which are being turned and tipped.

195. **Benches** on the sides of cuttings are small platforms, level transversely, seldom exceeding about six feet in breadth. They are sometimes used in very deep cuttings, for the purpose either of intercepting the fall of boulders and pieces of rock from the higher slopes, or of facilitating the drainage. A bench ought to have a slight declivity lengthwise, and at the foot of the slope above it there should be a side-drain like that at the side of the base (E, fig. 160, p. 334) to catch and carry away all the surface-water from that slope.

196. **Prevention of Slips.**—As the slipping of the sides of cuttings is caused by the action of water, its prevention is promoted by efficient ordinary drainage, as described in Article 189, p. 331, and Article 193, p. 335. When ordinary methods of drainage are insufficient, other expedients must be adopted, such as the following:—To make the branch drains of the slope very numerous and close; to make special drains for carrying down to the side-drain of the cuttings, the waters of such springs as may flow from the slope; to face the slope with a well-packed layer of stones laid dry, from 6 to 18 inches thick, according to the circumstances of the case; to cut away a portion of the lower part of the slope, and form in the space so left a bank of gravel or shivers of stone, against which the slope of earth may abut, with counterforts, made by digging trenches at right angles to the gravel bank, and filling them with gravel; this combination acts both as a retaining wall and as a system of drains; to build at the foot of the slope, so as at once to support and drain it, either a dry stone retaining wall, or a wall of brick or masonry laid in mortar, backed with a vertical layer of dry stones; to intercept underground waters on their way towards the slope, by means of a drift or mine.

Retaining walls will be further treated of under the head of MASONRY, and drifts under that of TUNNELLING.

In some instances, all remedies for slipping are found unavailing, and the material must be allowed to find its own angle of repose, care being taken to remove the earth which slides down from time to time, and to acquire the necessary additional land.

197. **Settlement of Embankments.**—Embankments subside, or settle after their first formation, to an extent which varies considerably for different materials and under different circumstances, being seldom less than *one-twelfth*, and seldom more than *one-fifth*, of the original height. The best method of ascertaining the probable

proportionate settlement of a proposed embankment is by an experiment on a short length of it, allowance for the settlement so ascertained must then be made in constructing the remainder of the embankment.

198. The **Distribution of Earthwork** means the arrangement by which the materials obtained from different parts of the cuttings are distributed amongst different parts of the embankments, so as to insure the least possible expenditure of labour in the *leading* or conveyance of the earth. To attain this object, two rules are to be followed as closely as may be practicable;—to fill each portion of embankment from the *nearest accessible* portion of cutting; and to take care that the several routes by which earth is conveyed from cutting to embankment shall not cross each other

The *mean distance* of lead, from a division of a cutting to that division of an embankment which is filled from it, is nearly equal to the distance between their centres of gravity.

199. **General Operations of Embanking**.—The best materials for embankments are those whose frictional stability is the greatest and the most permanent, such as shivers of rock, shingle, gravel, and clean sand Clay also forms safe embankments, provided it is dry, or nearly dry, when laid down. Wet clay, vegetable mould, and mud, are unfit for use in embankments; so also is peat, except with certain precautions to be afterwards mentioned.

An embankment may be made in three ways:—I. In one layer. II. In two or more thick layers. III. In thin layers.

I. *In one layer* —This being the cheapest and quickest method, consistent with stability, is that followed in all earthworks in which there is no special reason to the contrary In fig. 162, B A C represents the natural surface of the ground; D A part of the base of a cutting; A E C an embankment, the construction of which is carried forward in the direction A E of its full width and height (including a sufficient allowance for subsidence), by running earth-wagons on temporary rails from the cutting along the top of the embankment, and tipping them at E, so that the earth runs down and spreads itself over the sloping end E C of the bank, which is called the "tip."

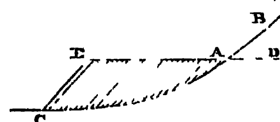


Fig. 162

The sloping lines parallel to E C represent a series of successive previous positions of the tip, as the embankment advanced from A.

No tipping over the sides of embankments should be allowed; for the earth so tipped is liable afterwards to slip off.

II. *In thick layers.*—This process has been used in some embankments of great height. It consists in completing the construction of the embankment up to a certain height by the process of

tipping over the end already described; leaving that layer for a time to settle, and then making a second layer in the same way, and so on. It involves much additional time and labour, and is seldom employed. It is, however, useful in making embankments of hard clay or shale, which, when first tipped, consists of angular lumps that lie with vacant spaces between them, and do not form a compact mass until partially softened and broken down by the action of air and moisture.

III. *In thin layers.*—This process consists in spreading the earth in horizontal layers of from 9 inches to 18 inches deep, and ramming each layer so as to make it compact and firm before laying down the next layer. Being a tedious and laborious process, it is used in special cases only, of which the principal are, the filling behind retaining walls, behind wings and abutments of bridges and culverts, and over their arches, and the embankments of reservoirs for water.

The *labour* of spreading earth in layers and ramming it may be estimated, in general, at from *once-and-a-sixth* to *once-and-a-third* that of shovelling it into a barrow. (See Article 194, p. 337.)

200. **Embanking on Sidelong Ground.**—When the natural ground has a steep sidelong slope, it is, in general, necessary to cut its surface into steps before making the embankment, in order that the latter may not slide down the slope. In the cross-section, fig. 163, the dotted line A B represents the natural surface of the ground, Q E B a side-cutting, and A D Q an embankment, resting on steps which have been cut between A and Q. The best position for those steps is *perpendicular to the axis of greatest pressure*, whose inclination to the vertical is given by equation 14 of Article 183, p. 322; so that, if A D is inclined at the angle of repose, the steps near A should be inclined to the horizon, in the opposite direction to A D, at the angle given by equation 16, p. 322; while the steps near Q may be level. It is better to make the steps steeper than the inclination given by this principle than to make them flatter.

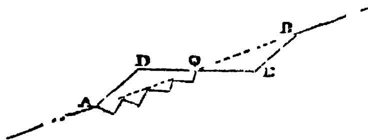


Fig 163

201. **Embanking over and near Masonry.**—In embanking over culverts (that is, covered drains of masonry or brickwork), near retaining walls, or near the abutment and wing walls of bridges, care must be taken not to injure the masonry by shocks from the fall of earth, or by ill-distributed or sudden pressures.

For the purpose of preventing shocks, the precaution is taken already mentioned in Article 199, above, of spreading the earth in

immediate contact with the masonry in thin layers, and ramming each layer. For this purpose, dry materials should be chosen, that will let water drain off easily, such as shivers of stone, gravel, and clean coarse sand. This rammed earth should fill all the space between the wing walls of bridges, and extend back from retaining walls, and from the abutments of bridges and culverts, ten feet or thereabouts. Over the arches of culverts, the earth rammed in thin layers should rise to at least half the height of the proposed embankment; the remainder may be tipped in the common way.

For the purpose of preventing unequal lateral pressures against bridges and large culverts, care should be taken (by the aid, if necessary, of timber platforms to carry the temporary rails), that the embankment is carried up at both sides of the structure at once, and as nearly as possible to the same height at the same time.

202. **Drainage of Embankments.**—The position and use of the catchwater drain near the foot of the slope has already been explained in Article 189, p. 334. The construction of culverts, for carrying drainage-water below embankments, will be treated of under the head of MASONRY. Ground in which springs rise should be avoided altogether, if possible, but if it is absolutely necessary to embank over a spring, a culvert may be built to carry its water clear of the embankment.

203. **Embankment in a Great Plain.**—When a line of conveyance is carried across an extensive plain, it is almost always necessary, in order to keep its surface dry, that it should be raised above the general level of the ground; and where inundations occur, the requisite height may be considerable. In fig. 164, A represents a

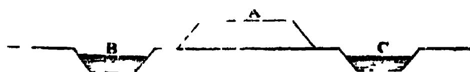


Fig. 164.

cross-section of an embankment for this purpose, the materials for which are obtained by digging a pair of trenches, B, C, alongside of it. These trenches, by collecting surface-water and discharging it into the nearest river or other main drainage channel, tend to shorten the duration of floods in the neighbourhood of the line.

204. **Embankments on Soft Ground.**—When the ground is so soft that an embankment made in the ordinary way would sink in it, different expedients are to be employed, according to the kind and degree of difficulty to be overcome. The following list of expedients is arranged in the order of an increasing scale of difficulty:—

I. By digging side-drains parallel to the site of the intended embankment, the firmness of the natural ground may be increased.

II. If the material of the natural ground has a definite angle of

repose, though much flatter than that of the material of the embankment, the slopes of the embankment may be formed to the same angle, thus giving it a broader foundation than it would have with its own natural slope.

III. A foundation may be made for the embankment by digging a trench and filling it with a stable material. In fig 165, A E F B represents the cross-section of an intended embankment, and A C D B that of the trench to be dug for its foundation, the edges of the base of the trench, C, D, being vertically below those of the top of the embankment, E, F. To design these cross-sections, proceed as follows:—

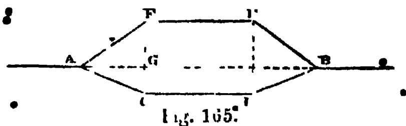


Fig. 165.

Let $h = $ G E denote the height of the proposed embankment;

w, the weight of a cubic foot of its material;

w', the weight of a cubic foot of the material of the natural ground;

ϕ', its angle of repose;

$h' = $ G C, the required depth of the foundation;

also let $\frac{1 - \sin \phi'}{1 + \sin \phi'} = k'$;

then the depth of the foundation is given by the formula,

$$h' = \frac{h w k'^2}{w' - w k'^2} \quad \ldots \ldots \quad (1)$$

The slopes of the trench, C A, D B, should be inclined at the angle of repose of the soft material; so that the breadth of each will be

$$\text{A C} = h' \cdot \text{cotan}\, \phi', \ldots \ldots \quad (2)$$

and this fixes also the inclination of the slopes of the embankment, A E, B F, without reference to the angle of repose of its material.

IV. The ground may be compressed and consolidated by means of short piles. This method will be further explained under the head of FOUNDATIONS.

V. The embankment may be made of materials light enough to form a sort of raft, floating on the soft ground, such as hurdles, fascines, or dry peat. The use of fascines will be further explained in a later chapter. Dry peat was the material used by George Stephenson to carry the Liverpool and Manchester Railway across Chat Moss. Its heaviness, when well dried in the air, is about

30 lbs. per cubic foot; and when saturated with water, 63 lbs. On the dry peat embankment was placed a platform of two layers of hurdles, to carry the ballast.

VI. Should all other expedients fail, a moss or bog may still be crossed by throwing in stones, gravel, and sand, until an embankment is formed, resting on the hard stratum below the moss, and with its top rising to the required level. It is found that the material of the embankment assumes the same natural slope that it would do in the air.

205. **Dressing, Soiling, and Pitching Slopes.**—The slopes, both of embankments and cuttings, are to be dressed to smooth and regular surfaces, and covered with a layer of soil, which varies from 3 inches to 6 inches in depth, according to the practice of different engineers, and is sown with grass-seed.

The *labour* of dressing slopes is nearly equivalent to that of digging about half-a-foot deep in loose mould over the same area of surface; and that of spreading the soil is about the same with that of shovelling it into a barrow. (See p. 337.)

Slopes of embankments which are exposed to still water may be faced or "pitched" with dry stone about a foot thick. The protection of slopes against waves and currents falls under the head of Hydraulic Engineering.

206. **Clay Puddle** is used to make embankments and channels water-tight, and to protect masonry against the penetration of water from behind. The proper material for it is clay, freed from all large stones, roots of plants, and the like, and containing as much sand and fine gravel as is consistent with its holding water; if there is too little sand, the puddle is liable to crack in dry weather. It is made by working the clay in layers about 9 inches thick, with enough of water to reduce it to a pasty condition, by means of a tool that has a sort of *poaching* action, until it becomes a perfectly uniform and compact mass. The labour is about five times that of shovelling the same quantity of material.

207. **Quarrying and Blasting Rock.**—Rock that is too hard to be split with the pick, the crowbar, or the quarryman's hammer, and not so hard as to require blasting with gunpowder, can be quarried in blocks, by cutting grooves, or boring holes, in the upper surface of a bed, inserting blunt steel wedges in them, and driving those wedges with a hammer until a block splits off from the layer. Gauthey's estimate of the labour of this operation, per cubic yard of rock, is about 0·4 of a day's work of a man; but it varies very much for different kinds of rock.

The processes of blasting with gunpowder may be divided into *small blasts* and *great blasts*.

I. A *small blast* is made by boring with a jumper (Article

187, p. 332), a hole in the rock, whose diameter varies from 1 inch to 6 inches, or thereabouts, and its depth from one foot to 30 feet. Part of the depth of the hole is filled with coarse-grained gunpowder, poured in through a tube reaching nearly to the bottom, and the remainder of the hole is rammed with what is called "tamping," consisting of chips of rock, sand, clay, and other such materials; the best material being dry clay. Care is to be taken never to use materials that may strike fire, and not to ram hard until there are some inches of material between the tamping bar and the powder. The fuse may be protected by traversing a tube, or a groove in a piece of wood. It should burn at the rate of about 2 feet per minute. The best fuse for this purpose is known as "Bickford's."

The explosion of the powder splits and loosens a mass of rock whose volume is approximately proportional to the *cube of the line of least resistance*,—that is, in general, of the shortest distance from the charge to the surface of the rock—and may be roughly estimated at *twice* that cube, but this proportion varies very much in different cases.

The proportion of the *weight of rock loosened* to the *weight of powder* exploded ranges from about 7,000 : 1 to 14,000 : 1, and may be taken on an average at 10,000 : 1.

The ordinary rule for the weight of powder in small blasts is,

$$\text{powder in lbs.} = \frac{(\text{line of least resistance in feet})^3}{32} \ldots \text{(1.)}$$

A test of the strength of blasting powder is, that 2 ounces, or $\frac{1}{8}$th of an avoirdupois pound of it, being fired in an eight-inch mortar elevated at an angle of 45°, should throw a 68 lb. ball to a distance of 240 feet.

Another test is by firing 2 ounces of powder in the "eprouvette gun;" its bore is 27·6 inches long, and $1\frac{3}{4}$ inch in diameter; it weighs $86\frac{1}{2}$ lbs; it is hung in a frame like a "ballistic pendulum," and its recoil is measured on a graduated arc. Good powder fit for blasting gives a recoil of about 20 degrees.

One lb. of powder in a loose state occupies about 30 cubic inches. By compression, it may be squeezed into $27\frac{1}{2}$, or thereabouts.

Thirty cubic inches are equal to 38·2 *cylindrical inches*; and this is the *length of hole, one inch in diameter, required to hold one lb. of powder*. The corresponding length for other diameters varies inversely as the square of the diameter.

A blast acts most efficiently when the line of least resistance (being, in sound rock of uniform strength, the shortest line from the charge to the surface), is perpendicular to the axis of the bore-

hole. It acts least efficiently when the line of least resistance is the axis of the bore-hole itself. It is not always possible to jump a hole perpendicular to the intended line of least resistance; but the hole should always be made to form as great an angle with that line as possible.

If a charge fails to explode, the tamping may be bored out with the auger (p. 332), a new fuse put in, and the hole re-tamped. This process, however, is not wholly free from danger; and the safest method is to jump a new hole near the first, and put in a fresh charge of powder, the explosion of which will probably be communicated to the former charge.

The labour of jumping holes varies very much for different qualities of rock. It is performed either by two men striking the jumper with hammers, while a man or boy turns it, or by one or two men raising it and letting it drop; the latter being the more efficient method, but wearing out the jumper faster than the other. The jumper used for the latter process is called the "churn jumper."

The following are examples of the day's work per man performed in jumping holes: —

	Cylindrical Inches of Hole
In granite, by hammering,	100 to 150
" by "churning,"	200 nearly
In limestone,	500 to 700

(As to jumping by machinery, see Article 392, p. 594.)

In granite, jumpers require to be sharpened about once for each foot bored, and steeled once for each 16 or 20 feet; and the length of iron wasted in using them is about one-tenth of the depth bored.

The lower ends of holes in limestone have sometimes been enlarged to form a chamber for the powder, by the aid of dilute nitric acid. A double tube, consisting of an outer tube of copper with a tube of lead within it, is passed down to the bottom of the hole; the inner tube has a funnel on the top into which the dilute acid is poured; it passes down, and dissolves the lime of the limestone; the carbonic acid gas disengaged forms, with the solution of nitrate of lime, a stream of froth, which rises through the space between the inner and outer tubes, and escapes through a lateral bent spout near the top of the latter.

II. A *great blast* is made by excavating a vertical shaft or a horizontal heading in the mass of rock, which should turn at right angles at least once on its way to the powder-chamber at its end, in order that the tamping may not be blown out. Such shafts and

headings vary from 3½ feet square to 3½ feet by 5 feet or thereabouts, and the labour required to make them varies from 2 days' to 6 days' work of a miner per lineal foot. The mine being swept out and its floor covered with a matting of old sacks, the gunpowder is placed in the chamber in a deal box, whose size is regulated by the fact that 1 lb. of gunpowder fills about 30 cubic inches; a small quantity of finer powder in a bag or case forms the "bursting charge," and is traversed by a fine platinum wire, connecting a pair of copper conducting wires with each other. These are coated with Indian rubber or gutta percha, or otherwise insulated, and protected by being placed in a groove in a wooden bar. The entrance of the chamber is closed with a wall of turf, and the rest of the mine "tamped" by being built up either with rubble masonry or with a mixture of stones and clay. When the workmen have removed to a safe distance, the conducting wires are connected with the opposite ends of a galvanic battery, when the electric current raises the platinum wire to a white heat, and fires the charge.

The chief use of the electrical apparatus is to fire several charges exactly at the same instant. When one charge only is to be fired, a safety fuse may be used.

According to Mr. Sim, the chamber of the mine should be so placed, that the line of least resistance may be about two-thirds of the height of the rock to be loosened.

In great blasts the proportion of the weight of the rock loosened to that of the powder exploded, ranges from 4,500 : 1 to nearly 13,000 : 1, and is on an average about 6,000 : 1 or 7,000 : 1.

The ratio of the number of lbs. of powder to the cube of the number of feet in the line of least resistance ranges from 1 : 32 to 1 : 10; but the best mode of fixing the quantity of powder is to estimate roughly the weight of the mass of rock which is likely to be loosened, and use from ½ to ⅓ of a lb. of powder for each ton of rock.

In choosing the positions of bores and mines for blasting, regard should be had to the natural veins and fissures of the rock, as means of facilitating its detachment from its bed.

Blasting under water will be considered in a later part of this treatise.*

* On the subject of blasting rock the following authorities may be consulted:—Guttman, *Blasting*; Foster, *Ore and Stone Mining*, 2nd Ed., 1897; Hughes, *Coal Mining*; Abel, "Explosive Agents applied to Industrial Purposes," *Minutes Proceed. Inst. C.E.*, vol. lxi.; "Removal of Obstructions at Iron Gate of the Danube" (see *The Engineer*, 1896, p. 335); *Minutes Proceed. Inst. C.E.*, vol. cxix.; also Berthelot, *Explosives and their Power*.

TABLE OF THE HEAVINESS OF ROCK.

	Lbs. in one Cubic Foot.	Lbs. in one Cubic Yard.	Cubic Feet to a Ton.
Basalt,	187	5060	12
Chalk,	117 to 174	3160 to 4730	19·1 to 12·9
Felspar,	162	4370	13·8
Flint,	164	4430	13·6
Granite,	164 to 172	4430 to 4640	13·6 to 13
Limestone,	169 to 175	4560 to 4720	13·2 to 12·8
" magnesian,	178	4810	12·6
Quartz,	165	4450	13·6
Sandstone, average,	144	3890	15·6
" different kinds,	130 to 157	3510 to 4240	17·2 to 14·3
Shale,	162	4370	13·8
Slate (Clay),	175 to 181	4720 to 4890	12·8 to 12·4
Trap,	170	4590	13·2

It is stated that to produce the same effect in blasting that is produced by a given weight of powder, *one-sixth* of that weight of blasting cotton, or *one-tenth* of that weight of blasting oil, is sufficient. Blasting oil (otherwise called "Nitroglycerine" or "Nitroleum") explodes by concussion; therefore it is dangerous to jump a new hole near a hole which is already charged with it.

Nitroglycerine has been rendered much more generally serviceable through its absorption by a siliceous earth found in different parts of Europe; in this form it is known as "Dynamite," for which see p. 796. There are various other combinations of nitroglycerine, such as cordite, blasting gelatine, and ballistite. Guncotton, referred to above, is a compound of nitric and sulphuric acids with cotton fibre Some of these compounds are smokeless, and can be fired although wet.

For authorities, see p. 347.

CHAPTER III.

OF MASONRY.

SECTION I.—*Of Natural Stones.*

208. **Structural Characters of Stones.**—In the last Article of the preceding chapter, rocks or natural stones were considered in the light of materials to be excavated. They have now to be considered in the light of materials for building.

The geological position of rocks has but little connection with their properties as building materials. As a general rule, the more ancient rocks are the stronger and the more durable; but to this there are many exceptions. The properties or characters of rocks which are of most importance in an engineering point of view are of two kinds; the structural and the chemical.

With respect to the structural character of their large masses, rocks may be divided into two great classes,—I. The unstratified, II. The stratified, according as they do not or do consist of flat layers.

I. The *Unstratified Rocks* are believed to have become solid more or less slowly, and under a greater or less pressure, from a melted state. They are, for the most part, hard, compact, strong, and durable.

It is in general obvious that the great masses of unstratified rocks are built, as it were, of blocks, which separate from each other when the rock decays. In granite, for example, those blocks are oblique hexaëdrons - in other words, rhomboidal prisms, sometimes of enormous size; in basalt, they are regular hexagonal or pentagonal prisms, built up into columns; in trap, they are irregular prisms, sometimes approximating imperfectly to the columnar form of basalt. In many cases the further progress of decay rounds off the corners and edges of the blocks, and converts them into boulders, which show a tendency to break up into concentric oval layers. In all cutting, quarrying, and blasting of unstratified rocks, the work is much facilitated by taking advantage of the natural joints between the blocks, at which the rock is more easily divided than elsewhere.

In their more minute structure the unstratified rocks present, for the most part, an aggregate of crystalline grains, firmly adhering together. In granite and syenite, these crystals are comparatively large and conspicuous; in trap, they are much smaller and less dis-

tinct; in basalt, they are almost invisible, and the structure is almost glassy; in lava, it is decidedly glassy. Amongst varieties of structure in unstratified rocks, are the porphyritic, where detached crystals of one substance are imbedded in a mass of another; and the cellular, where the mass contains a number of spherical or oval cavities, as if, in its former melted state, it had air-bubbles dispersed in it.

Masses of unstratified rock are often traversed by veins or cracks, sometimes empty, sometimes lined on the sides, and sometimes filled with crystalline masses of various minerals. Such veins facilitate the division of the rock where they traverse it.

II. *Stratified Rocks* consist of a series of parallel layers, evidently deposited from water, and originally horizontal, although in most cases they have become more or less inclined and curved by the action of disturbing forces. It is easier to divide them at the planes of division between those layers than elsewhere. They are traversed by veins or cracks, sometimes empty, sometimes containing crystals, sometimes filled with "dykes," or masses of unstratified rock. Those veins or dykes are often accompanied by a "fault," or abrupt alteration of the levels of the strata.

It is in the immediate neighbourhood of masses of unstratified rock that the stratified rocks show the greatest effects of the action of disturbing forces in the inclination, curvature, and distortion of their layers. In such positions, too, they often appear to have had their structure altered by heat and intense pressure, and to have been rendered harder and more compact.

Besides its principal layers or strata, a mass of stratified rock is in general capable of division into thinner layers; and although the surfaces of division of the thinner layers are often parallel to those of the strata, they are also often oblique, or even perpendicular to them. This constitutes a *laminated* structure. Laminated stones resist pressure more strongly in a direction perpendicular to their laminæ than parallel to them; they are more tenacious in a direction parallel to their laminæ than perpendicular to them; and they are more durable with the edges than with the sides of their laminæ exposed to the air; and, therefore, in building, they should be placed with their laminæ or "beds" perpendicular, or nearly so, to the direction of greatest pressure, and with the edges of these laminæ at the face of the wall.

In the more minute structure of stratified rocks the following varieties are distinguished:—

(1.) The *compact crystalline* structure, as in quartz rock and marble. This is accompanied by great strength and durability.

(2.) The *slaty* structure, when the rock, which is usually compact, can be split into innumerable thin layers, often highly inclined to

the stratification. This structure is considered to have arisen from intense pressure, in a direction perpendicular to the layers. It facilitates quarrying. Some of the stones in which it occurs, as hard clay-slate and hornblende-slate, are amongst the strongest and most durable known. Others are soft and perishable.

(3.) The *granular crystalline* structure, in which crystalline grains either adhere firmly together, as in gneiss, or are cemented together into one mass by some other material, as in sandstone. This is accompanied by various degrees of compactness, porosity, strength, and durability, from the highest to the lowest, passing at the lowest extreme into sand.

(4.) The *compact granular* structure, where the grains are too small to be visible, and seem to form a continuous mass, as in blue limestone. This structure is usually accompanied with considerable strength and durability. It passes by gradations on the one hand into the compact crystalline structure (1), and in the other into,

(5.) The *porous granular* structure, in which the grains are not crystalline, and are often, if not always, minute shells cemented together, as in oolite. The porosity of rocks having this structure varies much; and so also do the strength and durability, which are seldom very high. In these respects the lowest example is soft chalk.

(6.) The *conglomerate* structure, where fragments of one material are imbedded in a mass of another, as in grauwacke.

The *fracture*, or appearance of the broken surface of a stone, is one of the means of showing its structural character. The following are examples:—

The *even* fracture, when the surfaces of division are planes in definite positions, is characteristic of a crystalline structure.

The *uneven* fracture, when the broken surface presents sharp projections, is characteristic of a granular structure.

The *slaty* fracture is even for planes of division parallel to the lamination, and uneven for other directions of division.

The *conchoidal* fracture presents smooth concave and convex surfaces, and is characteristic of a hard and compact structure.

The *earthy* fracture leaves a rough dull surface, and indicates softness and brittleness.

209. **Chemical Constituents of Stones.**—The numerous substances which have not yet been decomposed, and which are therefore provisionally called "elementary substances" in chemistry, are all found in the composition of stones. These elementary substances form, by their combinations, a vast variety of compounds called "simple minerals," or "mineral species." Each simple mineral is a definite chemical compound, and is a homogeneous substance; that

is to say, every particle of it perceptible to any means of observation is similarly composed to every other. Most simple minerals are distinguished also by definite primary forms of crystallization. Two minerals which have the same chemical composition may still be distinguished as distinct species, by having different primary crystalline forms. Thomson enumerates more than 500 mineral species; Jameson gives about 110 genera, each containing from one to 10 species.

The masses which form the earth's crust, whether stony or earthy, stratified or unstratified, are made up of simple minerals, either of one kind or of several kinds, *mixed*, not chemically combined.

There are a few simple minerals which are so much more abundant in the earth's crust than the others, that they fix the predominant characters, both chemical and mechanical, of the stones into whose composition they enter; and those minerals alone, with their principal chemical constituents, need be considered in such a treatise as the present.

The principal chemical constituent of those *predominant minerals* are four EARTHS, viz.:—

I. *Silica*, or pure flint. Its chemical composition is (according to the British scale),—

One equivalent of silicon,.........................	28
Two equivalents of oxygen,.........................	32
One equivalent of silica,.........................	60

Silica exists uncombined in great abundance, in the form of quartz, sand, and flint. With other earths and alkalies it combines, acting as an acid. It is not soluble in any acid except the fluoric, nor when crystallized is it soluble in water; but by an indirect process it can be made to form a gelatinous compound with water.

II. *Alumina*, the base of clay. Its chemical composition is

Two equivalents of aluminium,..................	54·8
Three equivalents of oxygen,..................	48·0
One equivalent of alumina,..................	102·8

Alumina exists uncombined in the ruby and sapphire alone. In combination with other earths, it exists in great abundance. It acts either as an acid or as a base. It forms a paste with water; and by indirect processes, can be made to form a gelatinous compound with water.

III. *Lime* is thus composed,—

One equivalent of calcium,..................	40
One equivalent of oxygen,..................	16
One equivalent of lime,..................	56

Lime does not exist in nature uncombined; but in combination with carbonic acid and with other earths it is very abundant. It is strongly alkaline, and soluble to a small extent in water.

IV. *Magnesia* is thus composed—

One equivalent of magnesium,........................	24
One equivalent of oxygen,...	16
	40

Magnesia is not found in nature uncombined; in combination with carbonic acid and with other earths it is abundant, though not so much so as the three earths before-mentioned. It is alkaline, but not so highly so as lime, and is very sparingly soluble in water.

In some of the predominant minerals the two following ALKALIES are found, combined with earths. Their presence in stone promotes its decomposition when exposed to the weather:—

Names	Composition.	Equivalent.
V. *Potash*,............	Potassium 78·3 + oxygen 16	= 94·3
VI. *Soda*, ...	Sodium 46· + oxygen 16	= 62·

The following ACID exists abundantly in combination with lime and magnesia.

VII. *Carbonic Acid*,.. Carbon 12 + oxygen 32 = 44

The presence of carbonic acid in stones is made known by their effervescing when acted upon by stronger acids.

The metals iron and manganese also enter into the composition of the predominant minerals, in quantities comparatively small. Their chemical equivalents are, Iron, 56; Manganese, 55.

210. The **Predominant Minerals in Stones** are the following:—

I. QUARTZ is pure silica. Its heaviness is from 2·5 to 2·7 times that of water. Its primary crystalline form is a rhombohedron. Its most common external crystalline form is a regular six-sided prism, with a six-sided pyramidal summit.

When it occurs in transparent crystals, colourless or coloured, it is called *rock-crystal*. In a compact, translucent mass, it is called *hornstone*. In dark-coloured, translucent lumps, which are scattered through the chalk, it is called *flint*. In grains, or small crystals, more or less rounded at the edges and corners, it forms *sand*. There are various other forms of quartz, which it is unnecessary to mention. It is the most hard and durable of all the predominant minerals.

II. **Felspar** is a mineral genus of compounds of earths and alkalies, of which the three species whose composition is given below are the most abundant, especially the first. Their heaviness is from 2·5 to 2·8 times that of water.

1. *Common Felspar*, or *Potash Felspar*, is composed of silica, alumina, and potash, in proportions which nearly agree with the following constitution:—

6 equivalents of silica;
1 equivalent of alumina;
1 equivalent of potash.

2. *Soda Felspar* has the whole or part of the potash replaced by an equivalent quantity of soda.

3 *Lime Felspar* has the whole or part of the potash replaced by an equivalent quantity of lime.

Felspar, with a crystalline or compact granular structure, forms the white or flesh-coloured grains and crystals which are seen in granite, porphyry, and some other rocks to be afterwards mentioned. With a slaty structure, it forms *clinkstone*. With a soft granular structure and earthy fracture, it forms *claystone*. It presents all degrees of hardness and durability.

III. **Hornblende** presents great varieties in appearance and composition. Its heaviness is from 2·7 to 3·2 times that of water. The composition of the white variety agrees nearly with the following constitution:—

9 equivalents of silica;
6 equivalents of magnesia;
2 equivalent of lime;

and there is also a small quantity of fluorine, which may be combined with part of the calcium. The most common varieties are the dark-green and the black, in which part of the silica appears to be replaced by alumina, in the proportion of one equivalent of alumina for three of silica, and part of the magnesia by an equivalent quantity of protoxide of iron.

Dark-green or black hornblende forms a great part of the mass of greenstone or trap. It occurs in crystals, fibres, and grains, and has a glassy lustre, and a fracture sometimes conchoidal, sometimes uneven, sometimes slaty. It is one of the toughest and most durable of minerals.

IV. **Augite** much resembles hornblende in all its properties. The composition of its white varieties agrees nearly with the following:—

2 equivalents of silica;
1 equivalents of magnesia;
1 equivalents of lime;

while in the green and black varieties, part of the magnesia appears to be replaced by an equivalent quantity of protoxide of iron.

V. MICA is distinguished by having a laminated structure, so that it either consists of or can easily be split into transparent or semi-transparent layers or scales. It is flexible, and so soft that it can be cut with a knife. Its heaviness is from 2·8 to 3 times that of water. The composition of one variety is nearly as follows:—

15 equivalents of silica;
4 equivalents of alumina;
3 equivalents of potash;
5 equivalents of oxides of iron and of manganese.

In other varieties part of the potash would seem to be replaced by lithia, and by an additional quantity of oxides of iron and of manganese. Some kinds contain fluorine.

VI. CHLORITE, or green earth, is a compound of the silicates of magnesia, alumina, potash, and oxide of iron, with some water. It resembles mica in its laminated structure, and in its softness and flexibility. Its heaviness is from 2·7 to 2·8 times that of water. It occurs in small scales, in large sheets, and in slaty masses.

VII. CARBONATE OF LIME consists of one equivalent of carbonic acid and one of lime. It forms all the varieties of marble and limestone. These stones will be further described afterwards.

VIII. DOLOMITE is a compound of carbonate of lime and carbonate of magnesia, in the proportion of about two equivalents of the former to one of the latter. It forms various magnesian limestones, to be described further on.

211. **Stones Classed.**—The stones used in building are divided into three classes, each distinguished by the *earth* which forms its chief constituent. These are—

I. *Siliceous Stones.*
II. *Argillaceous Stones.*
III. *Calcareous Stones.*

212. **Siliceous Stones** are those in which silica is the characteristic earthy constituent. With a few exceptions their structure is *crystalline-granular*, and the crystalline grains contained in them are hard and durable; so that weakness and decay in them generally arise from the decomposition or disintegration of some softer and more perishable material, by which the grains are cemented together, or by the freezing of water in their pores, when they are porous.

The following are the principal siliceous stones used in building:—

I. GRANITE and SYENITE are unstratified rocks, consisting of quartz, felspar, mica, and hornblende. The name *granite* is specially applied to those specimens in which there is little or no hornblende;

the name *syenite* to those in which there is little or no mica; but both are popularly known as *granite*.

The quartz is in the form of clear, colourless or gray crystals; the hornblende (when present) in dark-green or black crystals; the mica in glistening scales, or grains composed of such scales; the felspar in compact opaque crystals, of a white, yellowish, or flesh colour.

Granite is found underlying the lowest or "primary" stratified rocks, and often rising through and over them in dykes, veins, and mountain masses, which naturally break up into large rhomboidal blocks, as stated in Article 208, p. 349.

The durability and hardness of granite are the greater the more quartz and hornblende predominate, and the less the quantity of felspar and mica, which are the more weak and perishable ingredients. Smallness and lustre in the crystals of felspar indicate durability; largeness and dullness, the reverse.

The best kinds of granite are the strongest and most lasting of building stones. The difficulty of working them, caused by their great hardness, is only overcome by long practice on the part of the stone-cutters. Minute ornaments cannot be carved in granite, and a simple and massive style of architecture is the best suited for it. It is used chiefly in works of great magnitude and importance, such as lighthouses, piers, breakwaters, and bridges over large rivers; and for such purposes it is brought from great distances at considerable cost, the stones being often cut to the required forms before leaving the quarry, with a view to save expense in carriage, and to obtain the benefit of the skill of stone-cutters accustomed to the material. It is only in districts where granite abounds that it is used for ordinary building purposes.

II. GNEISS and MICA SLATE consist of the same materials with granite, in a stratified form. They are found in the neighbourhood of granite, in strata much inclined, bent, and distorted, and often form great mountain masses. Gneiss resembles granite in its appearance and properties, but is less strong and durable. Mica slate is distinguished by containing little or no felspar, so that it consists chiefly of quartz and mica; it has a laminated or slaty structure, and the silky lustre of mica; it is a tough material, in directions parallel to its layers, but is more perishable than gneiss. Both these stones are used for ordinary masonry in the districts where they are found. Gneiss, from its stratified structure, is a good material for flag-stones. Mica slate, split into thin layers, may be used for covering roofs; but it is inferior for that purpose to clay slate.

III. GREENSTONE, WHINSTONE, or TRAP, and BASALT. These rocks are unstratified, and consist of granular crystals of hornblende

or of augite, with felspar. In greenstone the grains are considerably finer than in granite; in basalt they are scarcely distinguishable. Greenstone breaks up into small blocks; basalt into regular prismatic columns. (Article 208, p. 349.) They are found in veins, dykes, and tabular masses, amongst stratified rocks of various ages. Greenstone is usually dark-green, rarely white or red; basalt nearly black. These varieties of colour are due to the hornblende or the augite, the felspar being white. Both these rocks are very compact, durable, hard, and tough. The smallness of the blocks in which they can be obtained, and the difficulty of working them, prevent their being used in large works of masonry; but they are well adapted for ordinary building, and especially well suited for paving and metalling roads.

IV. Talc, Chlorite Slate, Soapstone. In these stones, silicate of magnesia predominates. *Talc* is in transparent or translucent sheets of a laminated structure; it is soft and easily cut. *Chlorite Slate* is also laminated, soft, and easily cut, but more opaque than talc; it is sometimes used for roofing, but is inferior to clay slate. It has a green or greenish-gray colour, and silky lustre.

Soapstone is translucent and soft, and greasy to the touch. It is valued for its power of resisting the action of fire.

V. Quartz Rock, Hornstone, Flint. These stones consist of quartz, pure, or nearly pure. *Quartz rock* and *Hornstone* are stratified, and appear to have been produced by the action of intense heat on standstone; they are both compact. Quartz rock is crystalline: hornstone is glassy. They are the strongest and most durable of all stones; but their hardness is so great as to make their use in masonry almost impracticable.

Flint is found in nodules or pebbles scattered through the chalk strata, and in beds of gravel, apparently left after the washing away of the chalk. It is hard and durable, but very brittle. Flints are used for building purposes by being made into a concrete with lime.

VI. Hornblende Slate is hard, tough, durable, and impervious to water, and is used for flag-stones.

VII. Sandstone is a stratified rock, consisting of grains of sand, that is, small crystals of quartz, cemented together by a material which is usually a compound of silica, alumina, and lime. In the strongest and most durable sandstone the cementing material is nearly pure silica; the weakest and least durable is that in which the cement contains much alumina, and resembles soft felspar or claystone. When there is much lime in the cementing matter of sandstone it decays rapidly in the atmosphere of the sea coast, and in that of towns where much coal is burned; in the former case the lime is dissolved by muriatic acid, in the latter by sulphuric

acid. *Calciferous sandstones*, as those containing much lime are called, pass by insensible degrees into sandy limestones. The appearance of strong and durable sandstone is characterized by sharpness of the grains, smallness of the quantity of cementing material, and a clear, shining, and translucent appearance on a newly broken surface. Rounded grains, and a dull, mealy surface, characterize soft and perishable sandstone. The best sandstone lies in thick strata, from which it can be cut in blocks that show very faint traces of stratification; that which is easily split into thin layers is weaker. Sandstone is found in every geological formation above the primary rocks, amongst which its place is supplied by hornstone and quartz-rock. The best kinds on the whole are those which belong to the coal formation; but they sometimes have their strength impaired by being divided into layers by extremely thin laminæ of coal.

The colours of sandstone are white, yellowish-red, and red, the latter colours being produced by the presence of peroxide of iron in the cementing material. Crystals of sulphuret of iron are sometimes imbedded in it; when exposed to air and moisture, they decompose, and cause disintegration of the stone. They are easily recognized by their yellow or yellowish-gray colour and metallic lustre. Sandstone is in general porous, and capable of absorbing much water; but it is comparatively little injured by moisture, unless when built with its layers set on edge, in which case the expansion of water in freezing between the layers makes them split or "scale" off from the face of the stone. When it is built "on its natural bed," any water which may penetrate between the edges of the layers has room readily to expand or escape.

The better kinds of sandstone are the most generally useful of building stones, being strong and lasting, and at the same time easily cut, sawn, and dressed in every way, and fit alike for every purpose of masonry.

213. **Argillaceous or Clayey Stones** are those in which alumina, although it may not always be the most abundant constituent, exists in sufficient quantity to give the stone its characteristic properties.

I. PORPHYRY consists of a mass of felspar, with crystals of felspar, and sometimes of quartz, hornblende, and other minerals, scattered through it. It occurs of all degrees of hardness. The variety in which the felspar matrix is soft and earthy is called *claystone porphyry;* it is of little or no value for building purposes. The hardest kind, in which the matrix is compact and crystalline, and the whole material beautifully coloured and capable of taking a high polish, is sometimes stronger than granite. It is rare, and is valued in building for ornamental purposes.

II. Clay Slate is a primary stratified rock of great hardness and density, with a laminated structure making in general a great angle with its planes of its stratification. (See Article 208, p.350.) Its colours are bluish-gray, blue, and purple, the darkest colours indicating in general the greatest strength and durability. It can be split into slabs and plates of small thickness and great area, and is nearly impervious to water; qualities which make it the best stony material for covering roofs, lining water-tanks, and similar purposes. The stronger kinds of clay slate have more tenacity along their laminæ than any other stone whose tenacity has been ascertained. The signs of good quality in slate are, compactness, smoothness, and uniformity of texture, clear dark colour, lustre, and the emission of a ringing sound when struck

III. Grauwacke Slate is a laminated claystone, containing sand, and sometimes fragments of mica and other minerals. It is used for roofing and for flag-stones, but is inferior to clay slate.

214. **Calcareous Stones** are those in which carbonate of lime predominates. They effervesce with the dilute mineral acids, which combine with the lime, and set free carbonic acid gas. Sulphuric acid forms an insoluble compound with the lime. Nitric and muriatic acid form compounds with it, which are soluble in water. By the action of intense heat the carbonic acid is expelled in the gaseous form, and the lime left in its caustic or alkaline state, when it is called *quicklime.* Some calcareous stones consist of pure carbonate of lime; in others it is mixed with sand, clay, and oxide of iron, or combined with carbonate of magnesia. The durability of calcareous stones depends on their compactness: those which are porous being disintegrated by the freezing of water, and by the chemical action of an acid atmosphere. They are, for the most part, easily wrought.

I. Marble is compact crystalline carbonate of lime. It is found chiefly amongst the primary strata, and generally in the neighbourhood of igneous rocks. It is translucent, capable of a fine polish, sometimes white, and sometimes variously coloured. It is one of the most durable of all stones. Its scarcity and value prevent its being used except for ornamental buildings.

II. Compact Limestone consists of carbonate of lime, either pure, or mixed with sand and clay. It varies in hardness and compactness, sometimes approaching to the condition of marble, sometimes to that of granular limestone. Its most frequent colours are white, grayish-blue, and whitish-brown. It is found amongst primary and secondary strata, and abounds specially in the coal formation, and in the lias formation. It is very useful as a building stone, and is durable in proportion to its compactness.

III. GRANULAR LIMESTONE consists of carbonate of lime in grains, which are in general shells or fragments of shells, cemented together by some compound of lime, silica and alumina, and often mixed with a greater or less quantity of sand. It is always more or less porous, and the less porous the more durable. It is found of various colours, especially white, and light yellowish-brown. In many cases it is so soft when first quarried that it can be cut with a knife, and hardens by exposure to the air. It is found in various strata, especially the oolitic formation. It there appears in the form of *Oolite*, or *Roestone*, so called because its grains are round, and resemble the roe of a fish. The pleasing colour and texture of oolite, and the ease with which it is wrought, have caused it to be much used in building, especially where delicate carving is required. The durability of oolites varies extremely. The Portland stone, the Bath stone, and the Aubigny stone (from Normandy) are examples of durable oolites. The perishable kinds of oolite decay more rapidly than almost any other stone, especially in an acid atmosphere.

IV. MAGNESIAN LIMESTONE, or DOLOMITE, is found in various conditions, from the compact crystalline to the porous granular. In Britain it is found in the new red sandstone formation immediately above the coal. It is like limestone in appearance. Its durability depends mainly on its texture; when that is compact it is nearly as lasting as marble, which it resembles in appearance; when porous it is very perishable.

215. **Strength of Stones.**—The external appearances from which the probable comparative strength or weakness of stones may be inferred have been stated in the course of the preceding articles.

Amongst stones of the same kind, that which has the greatest heaviness is almost invariably the strongest.

The results of past experiments on the strength of stones of the same kind differ very much from each other, probably owing to variations in the strength of the specimens of stone experimented on. The results given in the tables of the strength of materials at the end of the volume are averages from discordant data.

The following table contains some additional information on the resistance of stones to crushing, extracted from a paper which was read by Sir Wm. Fairbairn to the Manchester Philosophical Society, and published in his *Useful Information for Engineers*, second series :*—

* See also "Tables of Strength of Materials," p. 767.

	Crushing Stress, in lbs on the Square Inch.
Grauwacke from Penmaenmaur,	16,893
Basalt, Whinstone,	11,970
Granite (Mount Sorrel),	12,861
„ (Argyllshire),	10,917
Syenite, (Mount Sorrel),	11,820
Sandstone (Strong Yorkshire, mean of 9 experiments),	9,824
„ (weak specimens, locality not stated),	3,000 to 3,500
Limestone, compact (strong),	8,528
„ magnesian (strong),	7,098
„ „ (weak),	3,050

The experiments further show, that the resistance of strong sandstone to crushing in a direction parallel to the layers, is only *six-sevenths* of the resistance to crushing in a direction perpendicular to the layers.

The hardest stones alone give way to crushing at once, without previous warning. All others begin to crack or split under a load less than that which finally crushes them, in a proportion which ranges from a fraction little less than unity in the harder stones, down to about *one-half* in the softest.

The mode in which stone gives way to a crushing load is in general by *shearing*. (Article 157, pp. 235, 236; Article 158, p. 237.)

Experiments on the strength of stones have hitherto been made almost universally on cubical specimens. It is desirable that they should be made on prismatic specimens, whose heights are at least once and a-half their diameters; for an experiment made by crushing a cube indicates somewhat more than the real strength of the material.

When any building of importance is projected, the best course is not to trust to books for information as to the strength of the stone to be used, but to test it by special experiments, which can easily be made by the aid of a hydraulic press. As to the method to be followed in making those experiments, and calculating their results, in order to insure accuracy, see Article 144, pp. 223, 224.

The factor of safety in structures of stone should not be less than *eight*, in order to provide for variations in the strength of the material, as well as for other contingencies. In some structures which have stood it is less; but there can be no doubt that these err on the side of boldness.

216. **Testing Durability of Stones.**—The appearances which indicate probable durability have already been mentioned in describing particular kinds of stone; but they are often deceptive.

One test of the probable comparative durability of *stones of the same kind* is the smallness of the weight of water which a given weight of stone is capable of absorbing.

The following are examples:—

Granite absorbs one part of water in from	80 *to* 700	of stone.
Gneiss, " " " "	about 40	"
Clay Slate, " " " "	80 to 700	"
Sandstone (Strong, Yorkshire), "	30 to 60	"

Another test of probable comparative durability (invented by M. Brard) is to imitate the disintegrating action of frost by means of the crystallization of sulphate of soda, and weigh the fragments so detached from a block of a given size and surface in a given time. (*Annales de Chimie et de Physique*, vol. 38.)

The only sure test, however, of the durability of any kind of stone, is experience; and the engineer who proposes to use stone from a particular stratum in a particular locality in any important structure should carefully examine buildings in which that stone has been already used, especially those of old date.

The great difference which may exist in the durability of stones of the same kind, and presenting little difference in appearance, is strikingly exemplified at Oxford, where Christ Church Cathedral, built in the twelfth or thirteenth century, of oolite from a quarry about fifteen miles away, is in good preservation, while many colleges only two or three centuries old, built also of oolite, from a quarry in the neighbourhood of Oxford, are rapidly crumbling to pieces.

217. **Preservation of Stone.**—The various processes which have been tried or proposed for the preservation of naturally perishable stone all consist in filling the pores of the stone at and near its exposed surface with some substance which shall exclude air and moisture. In every case the surface of the stone should be prepared to receive the preserving material by expelling the existing moisture as completely as possible; and this is easily done by the aid of a portable furnace containing burning coke or charcoal.

The principal preserving materials are the following:—

Bituminous matter, such as coal tar. is very efficient; but unsightly from its colour. It is possible, however, that a colourless or light-coloured bituminous substance, suited for the preservation of stone, might be prepared by dissolving "paraffine" (so-called) in pitch-oil, or by some such process.

Drying Oil, such as linseed oil, either unmixed, or as an ingredient of paint, protects the stone for a time; but it is gradually destroyed by the oxygen of the air, so that it requires renewal from time to time; and it injures the appearance of the stone.

Silicate of Potash, or soluble glass, is applied in a state of solution in water, either alone or mixed with silica in fine powder. It gradually hardens, partly through the evaporation of its water, and partly through the removal of the potash by the carbonic acid of the air.

Silicate of Lime is produced by filling the pores of the stone with a solution of silicate of potash, and then introducing a solution of chloride of calcium, or of nitrate of lime. The chemical action of the two solutions produces silicate of lime, which forms an artificial stone, filling the pores of the natural stone, together with chloride of potassium or nitrate of potash, as the case may be, which salts, being soluble in water, are washed out.

The efficiency of the last two processes, and of various modifications of them, has of late been much contested. Time and experience only can show their real merits.

218. **Expansion of Stone by Heat.**—The following are the expansions in linear dimensions, according to the experiments of Mr. Adie, of some kinds of stone, when raised from the temperature of melting ice (32° Fahr.) to that of water boiling under the mean atmospheric pressure (212° Fahr.); that is, through 180° Fahr:—

Granite,	·0008 to ·0009
Marble,	·00065 to ·0011
Sandstone,	·0009 to ·0012
Slate,	·00104.

Section II.—*Of Bricks, and other Artificial Stones.*

219. **Clay for Bricks.**—The various sorts of clay, which are very numerous, are chemical compounds consisting of silicates of alumina, either alone, or combined with silicates of potash, soda, lime, magnesia, iron, and manganese. The complex clays approximate in their composition to felspar; and many of them may in fact be considered as soft varieties of felspar. (Article 210, p. 354.)

Clay and sand mechanically mixed constitute *loam;* clay and carbonate of lime mechanically mixed, *marl.* Amongst other substances which are found mixed with clay are peroxide of iron, sulphuret of iron, bitumen, &c.

Every kind of clay has the property, in its natural condition, of swelling and forming a paste when mixed with water. The expulsion of the water by heat is a slow process, and requires a high temperature, and is accompanied by shrinking and hardening of the mass of clay. It is doubtful at what temperature the ex-

pulsion of the water is complete; for so far as experiment has yet been carried, it appears that how high soever the temperature at which a mass of clay has been "burnt," as it is called, it will continue to shrink and to lose weight if raised to a higher temperature. A mass of burnt clay, at temperatures lower than that at which it has been burnt, expands with heat and contracts with cold like other solid substances.

By the operation of "burning," at a sufficiently high temperature, clay becomes hard and gritty, and loses either wholly or almost wholly the property of combining with water. Whether the clay afterwards slowly softens, and recovers that property or not, depends on its composition, and on the chemical agents to which it is exposed. The presence of alkaline constituents in the clay, and the action of acids upon it, tend to promote softening; and this goes on the more rapidly if it has been burned at too low a temperature.

Single earthy silicates, or compounds of silica with one other earth, are difficult of fusion, and resist the most intense heat of a furnace. This has been already exemplified in silicate of magnesia, or soapstone. (Article 212, p. 357.) Double, or more complex silicates, are more easily fusible, especially if one of the two or more silicates that are combined has for its base potash, soda, or lime. In conformity with this general law, the *refractory clays*, or those which resist fusion by the greatest heat of an ordinary furnace, are those which consist of silicates of alumina alone; and such clays only are fit to make fire-bricks and crucibles, and to cement together the parts of furnaces.

The following are examples:—

Porcelain Clay, or *Kaolin*, consists of

2 equivalents of alumina,
3 equivalents of silica,

which compound, in the natural state, is combined with two equivalents of water, nearly all of which can be expelled by a white heat. It is found in the neighbourhood of granitic rocks, having been formed by the slow decomposition of potash-felspar, under the action of the carbonic acid and moisture of the atmosphere, which have abstracted the potash and nine equivalents of silica. Its colour is white or cream-colour.

Stourbridge Fire Clay consists of

1 equivalent of alumina,
3 equivalents of silica,

with two equivalents of water, or thereabouts, which can be nearly all expelled by a white heat, and a small quantity of oxide of iron.

This and other fire-clays are found chiefly in the coal formation. Their colours are white, light-gray, and yellowish-gray, the colouring matter being in general a small quantity of oxide of iron.

Common Clays are rendered less difficult to fuse than porcelain clay and fire-clay, by the presence of silicates of lime, magnesia, and protoxide of iron; and the bricks made of them, when thoroughly burned, are partially vitrified. Of those constituents, protoxide of iron is the most favourable to the quality of the clay as regards the purpose of brickmaking, as it promotes the strength and hardness of the bricks. Its presence is shown by a dark greyish-blue colour, which is changed to red at and near the surface of the bricks by burning.

Silicate of lime in the clay in any considerable quantity makes it too fusible, so that the bricks soften in the kilns and become distorted.

Carbonate of lime, mixed with the clay in considerable quantity (indicated by effervescence with acids), loses its carbonic acid during the burning; and the quicklime which remains tends afterwards to absorb moisture, and cause disintegration of the brick. Clay containing this impurity should be avoided in making bricks.

Sand mixed with the clay in moderate quantity is beneficial, as tending to prevent excessive shrinking in the fire. Excess of sand makes the bricks too brittle. One part by volume of sand to four or five of pure clay is about the best proportion.

220. **Manufacture and Qualities of Bricks.**—In making bricks, the clay having been cleared of stones, is "tempered;" that is, mixed with about half its volume of water, and worked by stirring and kneading until it forms a perfectly uniform and homogeneous paste. The quality of the bricks depends mainly on the efficiency with which this is done. It may be performed by a machine called a "*pug-mill,*" in which the clay, contained in a vertical cylinder or barrel, is stirred and mixed by flat arms projecting from a rotating vertical axis, and at the same time forced downwards by the obliquity of the surfaces of those arms, so as to be made to stream slowly from a hole near the lower end of the barrel.

The wet clay, having been properly tempered and worked, is formed into bricks in moulds, which are larger than the bricks are intended to be when burned, by about 1-10th or 1-12th of each dimension, that being the ordinary proportion in which the dimensions of the brick shrink in burning.

Ordinary moulds for bricks measure about 10 inches in length, 5 inches in breadth, and 3 inches or thereabouts in depth; but bricks for special purposes are moulded of a great variety of shapes. *Various machines have been invented for moulding them.*

The bricks, having been dried in the open air, or in a drying house with a temperature of from 50° to 70°, are burned in kilns whose temperature is gradually raised in the course of twenty-four hours to a white heat, maintained nearly at that temperature until the bricks are sufficiently burnt, and then allowed to cool by slow degrees. The duration of the process of burning and cooling varies; but fifteen days, or thereabouts, is not unusual.

From data contained in a paper by Mr. F. W. Simms, it appears that the labour of making bricks by hand is as follows:—

Three men, viz.,.... { one temperer, one moulder, one wheeler,

and two boys, viz., { one carrier boy, one picker-up boy,

make 16,100 bricks per week. This is at the rate of

1·12 days' work of a man, and } per 1,000 bricks.
0·75 days' work of a boy, }

The fuel consumed in burning bricks ranges from 5 to 10 cwt. per 1,000 bricks.

The following are characteristics of good bricks.

To be regular in shape, with plane parallel surfaces and sharp right-angled edges.

To give a clear ringing sound when struck.

When broken, to show a compact uniform structure, hard and somewhat glassy, and free from air-bubbles and cracks.

Not to absorb more than about one-fifteenth of their weight of water.

Bricks which answer the preceding description, when *set on end* in a hydraulic press, should require at least 1,100 lbs. on the square inch to crush them, agreeably to the strength of "strong red bricks," as stated in the table at the end of the volume; and they will sometimes bear considerably more. A *small pillar of brick-work*, made of bricks of this quality laid in cement, should require from 800 to 1,000 lbs. on the square inch to crush it. (See page 237.)

Bricks in general begin to show signs of giving way, by splintering and cracking, when under about one-half or two-thirds of their crushing load.

The weaker qualities of bricks may be estimated as having from one-half to two-thirds of the strength stated above.

The bricks supplied for every building of importance should be carefully inspected, and the defective ones thrown away.

The expansion of bricks by heat, in rising from 32° to 212° Fahr. is as follows, according to Mr. Adie:—

Common brick, ...	·00355
Fire brick,...	·0005.

221. **Compressed Bricks** are made by drying the clay, grinding it to a fine powder, putting it into moulds of proper shapes, subjecting it to a pressure of about 5 tons on the square inch, and baking the bricks in a pottery-oven. The bricks so made have about once and a-half the heaviness of ordinary bricks, and considerably greater strength. They shrink very little in baking.

222. **Other Artificial Stones.**—Artificial sandstones, closely resembling natural sandstone in appearance, strength, and durability are made by cementing clean sharp sand together with silicate of potash (Kuhlmann's process), or silicate of lime (Ransome's process)

In the latter case, clean sharp sand is made into a paste with silicate of soda, and moulded into blocks, which are immersed in a solution of chloride of calcium; the latter substance penetrates the whole block, producing silicate of lime, which cements the sand together, and chloride of sodium, which gradually escapes in solution.

(As to *Concrete*, see pp. 373 and 793.)

SECTION III.—*Of Cementing Materials.*

223. **Analysis of Limestones and Cement Stones.**—Stones containing carbonate of lime in combination and mixture with other minerals are the most abundant and useful source of the cementing materials used in masonry. The following are their principal constituents, with their chemical equivalents:—

Carbonic Acid (see p. 353), ...	44	
Lime (see p. 352), ...	56	
Carbonate of Lime, 44 + 56 = ...		100
Magnesia (see p. 353), ...	40	
Carbonate of Magnesia, 44 + 40 =		84
Silica (see p. 352), ...	60	
Alumina (see p. 352), ...	102·8	
Ferrous oxide of iron (see p. 495)	72	
Ferric oxide of iron (iron 112 + oxygen 48),	160	
Water (hydrogen 2 + oxygen 16),	18	

It would be out of place in this work to enter into details of chemical processes; nevertheless, it may be useful to give the following directions for determining roughly the proportions of those constituents of limestone which are of the greatest practical importance.

I. Weigh a specimen carefully; calcine it in a crucible, and weigh it again; the loss of weight shows the quantity of *carbonic acid* and *water* together in the specimen; but if it has been well dried previously, at a temperature not sufficient to expel carbonic acid (which requires a bright red heat), the water remaining may be neglected, and the whole loss considered as *carbonic acid*.

II. Weigh another specimen of from 30 to 80 grains; reduce it to impalpable powder in a mortar, mix it with three times its weight of caustic potash or soda, and heat it to redness in a silver crucible, dissolve the whole in slightly diluted muriatic acid; the rapidity of solution may be increased by heating the diluted acid to near the boiling point of water. Evaporate the solution, taking care to stir it continually towards the end of the process, until it becomes thick and pasty: this shows that the silica has coagulated; mix the paste with eight or ten times its volume of boiling water:—this will dissolve every constituent except the silica:—filter the solution, washing the precipitate well with water, taking care to preserve all the water so used along with the original liquor, dry and calcine the precipitate left on the filter; weigh it:—this will give the quantity of *silica* in the specimen.

III. To the liquor add water of ammonia in excess; to precipitate the *alumina*, the *oxide of iron*, and *part of the magnesia*.

Then add lime-water by degrees as long as a precipitate falls. That precipitate is the *remainder of the magnesia*. Wash the whole precipitate; dry it; calcine it; weigh it. To the weight thus found add the weight of the silica found by operation II, and that of the carbonic acid as calculated from the result of operation I, subtract the sum from the whole weight of the specimen; the remainder will be the *lime*.

IV. From the total carbonic acid found by process I, (and reduced to the weight of the second specimen) subtract the weight of lime found by process III. $\times 44 \div 56$; the remainder will be the quantity of carbonic acid in combination with magnesia; and that remainder $\times 40 \div 44$ will give the quantity of *magnesia in combination with carbonic acid*.

V. Subtract the result of process IV. from that of process III.; the result will be the quantity of *alumina and oxide of iron*, and of *magnesia combined with silica*, if any; but in fact, so far as limestones are known, the whole of the magnesia is in the state of carbonate. For the present purpose it is unnecessary to separate the alumina from the oxide of iron (although it might be done by means of caustic potash, which dissolves the alumina and leaves the iron).

The most important result of the analysis is the proportion of *carbonates* to *silicates* in the stone. The quantity of carbonates

may be approximated to in a rough way by multiplying the total quantity of carbonic acid, as found by the first process, by the following multipliers:—

If the limestone is not magnesian,.................. 2·3;
if there is one equivalent of carbonate of magnesia for each equivalent of carbonate of lime, 2·12;

and the truth will almost always be between those limits. The remainder of the stone may be held to consist wholly or almost wholly of silicates.

The substances obtained by calcining different limestones and cement stones may be divided into the following four classes:—

I. *Pure, Rich,* or *Fat Lime,* produced from stones containing little or no silicate, which "slakes" by absorbing moisture, and having been made into a paste with water, hardens slowly in air, and not at all under water.

II. *Hydraulic Limes,* produced from stones containing moderate quantities of silicates (from 10 to 30 per cent), which slake, but less rapidly than pure lime, and harden under water slowly. These pass by insensible gradations into

III. *Cements,* produced from stones containing from 40 to 60 per cent. of silicates, which do not slake, and which harden quickly under water.

IV. *Pozzolanas,* which contain silicates in excess, and are used to make cement by mixing them with pure lime

224. **Pure, Rich, or Fat Lime** is made by calcining, at a bright red heat or somewhat higher, limestone that consists wholly or almost wholly of carbonate of lime, such as marble, or chalk. Such limestone loses 44 per cent. of its weight by burning, and leaves 56 per cent. of its weight of lime. Of this, about one-eighth is usually wasted.

The operation of lime burning is performed in kilns of two sorts. In the more common kind, the kiln is circular in plan, and oval in vertical section, the diameter at the bottom being about 6-10ths of the greatest diameter; it seldom exceeds 10 or 12 feet in height, but may be less; it is filled with alternate layers of limestone and fuel, and when the burning is completed the whole charge is removed. The whole operation takes from 30 to 50 hours. Another sort of kiln is cylindrical, or nearly so, with its axis vertical. It is continuously fed with limestone at the top, which descends, and is calcined by the flame coming from a furnace at one side of the kiln, and reaches the bottom completely burnt, whence it is gradually removed.

The weight of the coal consumed is from 1-5th to 1-6th of that of the lime burned.

One cubic foot of chalk in block, weighs 90 lbs. nearly.
" of broken chalk,............ 63 "
" of the quicklime from the same chalk, in pieces, 35 "

When rich quicklime is moistened with water, it *slakes;* that is to say, it combines chemically with one equivalent of water (9 parts by weight) to one equivalent of lime (28·5 parts by weight), and forms *slaked lime*, or in chemical language, *hydrate of lime.* During this process, the lime swells to from twice-and-a-half to thrice-and-a-half its original bulk, becomes very hot, and falls to powder. The same process takes place slowly through absorption of moisture from the atmosphere; but lime for building purposes ought never to be "air-slaked," as this slow operation is called; for the lime thus exposed to the air absorbs not only water, but carbonic acid; and part of it returns to the state of carbonate of lime. To guard against this sort of deterioration, quicklime should be kept in barrels, or in a dry store, until it is required for use, and then rapidly slaked with water. The hardening of slaked lime is produced by gradual absorption of carbonic acid from the atmosphere, and crystallization of the carbonate of lime so formed. It is a very slow process, but produces, after the lapse of years, a very hard material.

225. **Hydraulic Limes** are obtained by burning limestones, which contain silicates of alumina, and sometimes carbonate of magnesia, and which in general are compact, and of a gray, blue, or brownish-yellow colour. Besides the test of chemical analysis already mentioned in Article 223, the following direct test may be applied to limestones supposed to be hydraulic.

Calcine two or three cubic inches of the stone in a crucible:—pound the calcined lime: make it into a stiff paste with water, and form it into a ball, which immerse in a glass of water. If it is *hydraulic lime*, it will harden under water so as to resist the pressure of the finger in a time varying from 24 hours to a fortnight, according to its composition; and if its quality is good, in a month it will be about as hard as weak limestone.

If it is *cement*, it will harden so as to resist the pressure of the finger in a few minutes.

The best kinds of hydraulic lime slake so imperfectly, that they must be pulverized by grinding them in the dry state in a mill consisting of a circular trough, in which two stone rollers shaped like millstones, at opposite ends of a bar rotating with a vertical shaft, roll round, and so crush and grind the lime to a fine powder. Hydraulic lime should be kept in sacks or barrels in a dry store, and exposed as little as possible to air and moisture until it is about to be used.

It is a mixture of quicklime with silicates of alumina and iron, and sometimes with magnesia. Its hardening under water arises from the formation of an artificial stone, consisting of compound silicates of lime, alumina, and the other bases. In hydraulic lime, as distinguished from cement, there would seem to be a greater or less surplus of lime beyond that which is capable of combining with the silica and alumina. (See p. 804.)

226. **Natural Cement** is obtained by burning stones in which carbonate of lime and silicates exist in such proportions that, when the carbonic acid is expelled, the lime is exactly in the proportion required to make a hard compound with the silica and alumina. From the experiments of M. Vicat and of General Sir Charles Pasley, on making artificial cements, it would appear that the best mixture for making cement consists, before burning, of

two equivalents of carbonate of lime,... 100 × 2 = 200
one equivalent of clay, of which the probable composition is

one equivalent of alumina,......	102·8	
two equivalents of silica,.........	120·	222·8
		422·8

so that the composition in one hundred parts is,

carbonate of lime,....................	47
clay,........................	53
	100

and the rapidity with which the cement hardens under water depends on the nearness with which the composition of the stone approximates to these proportions.

Cement stones are usually found in thin strata amongst those of hydraulic limestone. Their most frequent colours are brown and fawn-coloured, their texture compact, and fracture earthy. After having been burned, they are ground to powder, which is packed in barrels, and carefully kept dry till required for use. In this state, it consists of a *mixture* of quicklime with silicate of alumina. So soon as it is made into a paste with water, chemical action takes place, and a double silicate of alumina and lime is formed, whose composition, in the best cement, would seem to be,

two equivalents of lime,..............	56 × 2 =	112·
one equivalent of alumina,...........		102·8
two equivalents of silica,............	60 × 2 =	120
		334·8

and this double silicate forms a compact artificial stone. (See p. 808.)

227. **Artificial Cement** is made by taking either ground chalk or slaked pure lime, and blue clay, in the proportions that will give the chemical composition stated in the preceding article, thoroughly mixing those ingredients into a paste with water in a pug mill, making the paste into balls of 2 or 3 inches in diameter, drying those balls, calcining them, and grinding them to powder. It is equal, if not superior, to any natural cement.*

228. **Pozzolanas** are mixtures analogous to cements, but containing an excess of silicates and a deficiency of lime, so that they must be mixed with pure lime in order to make cement, or hydraulic lime, according to the proportions. Amongst the best of these are iron scale and mine-dust, which consist of silicates of alumina and iron. If mixed with lime, so as to give the mortar a gray colour, they produce cement of extraordinary hardness and tenacity, which is probably a treble silicate of lime, alumina, and iron.

An ordinary proportion of such materials is about one part (by volume) to two parts of hydraulic lime, measured in the dry state; but the best way to fix the proportions is by trial.

Artificial pozzolana may be made by grinding bricks, or by burning good brick-clay and grinding it; in short, by any process which yields a dry powder of silicate of alumina, or compound silicate of alumina and iron.

229. **Mortar, Common and Hydraulic.**—Mortar is made by mixing lime and sand with enough of water to form them into a semi-fluid paste, in which state it is used as a binding material in masonry and brickwork.

Common Mortar, being made with pure lime, hardens slowly by the evaporation of the water, and by the absorption of carbonic acid from the atmosphere, forming crystalline carbonate of lime. If the water evaporates too fast, the mortar falls to powder; if it does not evaporate, the mortar remains always soft. Very slow evaporation of the water is therefore favourable to the ultimate hardness of the mortar.

Hydraulic Mortar, being made with hydraulic lime, hardens partly by the formation and crystallization of carbonate of lime, as above stated, but principally by the formation and crystallization of a complex silicate of lime, alumina, and other bases. (See Article 225, p. 371.)

Common mortar may be made hydraulic by a mixture of pozzolana. (Article 228, above.)

The sand employed should be clean, sharp, and rather coarse than fine. In order to render sand which is naturally mixed with clay fit for use in making mortar, it should be washed by stirring it amongst water, a slight current of which will carry away the

* For authorities on cements, see p. 436, also Appendix, p. 807.

in suspension, and leave the sand. Good sand for mortar is obtained by crushing soft sandstone. Sea-sand should be washed with fresh water; otherwise the salts contained in it will keep the mortar always moist.

In hydraulic as well as in common mortar the sand remains in a state merely of mechanical mixture, so that the mortar, when hardened, becomes a sort of artificial sandstone, consisting of grains of sand imbedded in a matrix of carbonate of lime, or of silicate of lime and other bases, as the case may be. The uses of the mixture of sand with the lime are as follows:—

To save expense, by diminishing the bulk of the lime, which is the more costly material, required to fill a given joint in the masonry.

To increase the resistance of the mortar to crushing.

To lessen the amount of shrinking, and the consequent tendency to crack, during the drying of the mortar.

But, at the same time, the mixture of sand diminishes the tenacity of the mortar; and if too much be used, the mortar will become brittle, and fall to powder as it dries.

The proportion of sand which lime will "bear," as it is called, without making the mortar brittle, is the greater the purer the lime, and the less the more strongly hydraulic the lime is. The best proportions, according to Vicat, are—

2·4 measures of sand to 1 of pure slaked lime in paste;
1·8 measures of sand to 1 of good hydraulic lime in paste;

and lime of intermediate qualities bears intermediate proportions of sand.

When sand and pozzolana are mixed with pure lime to make hydraulic mortar, the sand and pozzolana together may measure from twice to three times the volume of the lime before slaking.

In mixing mortar, however, the best method is, to ascertain the proper proportions in each case by trial.

The *labour of mixing mortar by the shovel* may be estimated at about

¾ of a day's work of a man per cubic yard.

A *two horse pug-mill* mixes mortar at the rate of from 20 to 25 cubic yards per day.

As hydraulic mortar tends to *set*, or harden, even in the wet state, it should not be mixed until immediately before it is required for use.

230. **Concrete and Beton.**—Common concrete is a mixture of mortar with gravel, in proportions such that the gravel and sand together are about six times the volume of the lime. It may be mixed either by hand or by the pug mill. (See p. 793.)

Strong Hydraulic Concrete, to which the name *beton*, borrowed from the French, has been applied, is made by mixing angular fragments of stone of from $1\frac{1}{2}$ to 2 inches in diameter, with hydraulic mortar in such proportions, that the mortar is a little more than enough to fill the spaces between the stones:—a proportion which is easily found by trial in each case. It may, however, be estimated as varying from

1 volume of stones and 1 volume of mortar, to
2 volumes of stones and 1 volume of mortar.

Concrete and beton, when mixed, occupy at first from two-thirds to three-fourths of the total volume of their materials before mixing; and when laid and rammed, they undergo a further settlement of about one-sixth; so that the final volume of concrete and beton varies from

$\frac{5}{9}$ to $\frac{5}{8}$ of the volume of the materials when unmixed.

When rounded stones only can be obtained, such as flints from the chalk strata, they may be made fit for the composition of beton by breaking them with the hammer into angular pieces. (See p. 436.)

231. **Mixed Cement.**—Cement which is to dry and set when fully exposed to the air, as at the outer edges of joints, or on the face of a wall, should be mixed with sand, to prevent unequal drying, and consequent shrinking and cracking. The proportions vary from

1 measure of sand and 2 of cement, to
1 measure of sand and 1 of cement.

Every mixture of sand diminishes the tenacity of cement; so much so that a mixture of equal parts of sand and cement has only one-fourth part of the tenacity of pure cement. Where the surface of the cement, therefore, is not exposed to the air, the only advantage of a mixture of sand is the saving of expense; and where great tenacity is required, pure cement should be used.

232. **Strength of Mortar, Cement, Concrete, and Beton.**—To the data given in the tables in the appendix, the following results of experiment may be added:—

A Year and a-half after Mixture.	Crushing Force in lbs. on the Square Inch
Mortar of Lime and River-Sand,	440
,, ,, ,, beaten,	600
Mortar of Lime and Pit-Sand,	580
,, ,, ,, beaten,	800
Hydraulic Mortar, of lime and pounded tiles,	680
,, ,, ,, beaten,	930
Beton, or concrete, of mortar and broken flints,	420

SIXTEEN YEARS AFTER MIXTURE, the increase of strength is in the following proportions:—

For common mortar,	1-8th
For hydraulic mortar,	1-4th.

(The above results are given on the authority of Rondelet.)

ONE YEAR AFTER MIXTURE.	TENACITY in lbs. on the Square Inch.
Good hydraulic lime,	170
Ordinary hydraulic lime from	140
Ordinary hydraulic lime to	100
Rich lime,	40
Good hydraulic mortar,	140
Ordinary hydraulic mortar,	85
Good common mortar,	50
Bad common mortar,	20

(The above are from Vicat.)

SIX MONTHS AFTER MIXTURE.	
Adhesion of common mortar to compact limestone,	15
Adhesion of common mortar to brick,	33

(The above are from Rondelet.)

Cement from Chalk Lime and Blue Clay, a few days after mixture (Sir C. W. Pasley),	125
Portland Cement (from compact limestone and clay) 30 to 50 days after mixture,	1200 to 1550

233. **Gypsum—Plaster of Paris.**—Gypsum is a compound of sulphate of lime with water, in the following proportions:—

One equivalent of sulphuric acid (sulphur 32 + oxygen 48) =	80
One equivalent of lime,	56
Two equivalents of water,	36
	172

It is found stratified, and in various conditions, crystalline, laminated, granular, and earthy. It is translucent, usually white or gray, has a pearly lustre, and can be easily scratched with a knife, being intermediate in hardness between rock-salt and calcareous spar.

By calcining gypsum the water is expelled, and it becomes a

dry white powder of sulphate of lime, known as "*Plaster of Paris.*" When this powder is rapidly mixed with water, so as to form a paste, it immediately begins to combine with part of the water, so as to reproduce gypsum in a compact granular state; heat is at the same time developed, which hastens the evaporation of the superfluous water. The mixture should be made by putting the powder into the water, not the water amongst the powder. The proportion of water used varies according to the purpose to which the plaster is to be applied; on an average it is about equal in bulk to the powder.

The tenacity of plaster, after it has "set," or hardened, according to Rondelet, is about 70 lbs. per square inch.

234. **Bituminous Cement and Concrete.**—A bituminous cement is a mixture of a pitchy or bituminous substance with an earthy substance.

For example, *Asphaltic Mastic* is made by mixing *Bitumen*, or mineral tar, obtained from bituminous shale or sandstone, with the powder of bituminous limestone or *asphalt*:—a mineral which consists of carbonate of lime, containing in its pores from 3 to 15 per cent. of bitumen.

The asphalt may either be broken into small fragments or ground to powder. It is then combined with the bitumen by heating the latter in an iron boiler, and adding the asphalt by degrees, taking care to mix the ingredients well. The proportions vary with the composition of the asphalt, less bitumen being required for that asphalt which contains much bitumen. The average proportion may be estimated at about 1 part by measure of bitumen to 7 or 8 of asphalt.

Artificial asphaltic mastic may be made by substituting coal-tar, or a solution of pitch in pitch-oil, for bitumen, and adding to it finely ground limestone till a proper consistency is attained.

A mastic composed of coal-tar and finely ground fireclay, in proportions which have never been exactly determined, but which are adjusted by trial until the mixture when cool is just soft enough to yield visibly to the pressure of the nail, and no more, has been found exceedingly useful to make tight joints in pipes, especially those traversed by strong acids, which would act upon a mastic containing limestone.

A bituminous or asphaltic mortar, as it may be called, is made by adding to the before-mentioned mixture of bitumen and powdered asphalt, about a thirty-fifth of its bulk of resin oil, and three-fifths of its bulk of sand.

11 measures of this mixture, with 9 of broken stone, make a sort of *bituminous concrete*, suited for covering the surfaces of roads, and for building under water.

Section IV.—*Of Ordinary Foundations.*

235. **Ordinary Foundations Defined and Classed.**—The *foundation* of a work of masonry on land consists, in the first place, of an excavation in the ground, and secondly, if required, of a structure at the bottom of that excavation, suited to form a firm base for the masonry. The foundations to which this section relates are those in which either an excavation alone is required, or an excavation partially filled with sand, stones, concrete, or beton. Foundations of a more difficult character, and requiring more complex works to render them secure, will be treated of in a later chapter.

Ordinary foundations are ranged under three classes, viz.:—

I. *Foundations in Rock*, or material whose stability is not impaired by saturation with water.

II. *Foundations in firm earth*, such as sand, gravel, and hard clay.

III. *Foundations in soft earth.*

The base of every foundation should be as nearly as possible perpendicular to the direction of the pressure which it is to sustain, and of sufficient area to bear that pressure with safety. The area is increased to any required extent by making the lowest courses of masonry or brickwork in the building spread out by a series of steps; by supporting them on a sufficiently broad layer of concrete or beton; by making inverted arches under openings; and by other contrivances. The *centre of resistance* of the foundation of a piece of masonry (or point traversed by the resultant of the pressure), should not deviate from the centre of gravity of its figure beyond certain limits, which will be afterwards specified in particular cases.

236. **Rock Foundations.**—To prepare a rock foundation for being built upon, the following are in general all the operations that are required:—

I. To cut away all loose and decayed parts of the rock.

II. To cut and dress the rock to a plane surface, or to a set of plane surfaces like those of steps, perpendicular, or nearly perpendicular, to the pressure to be sustained.

III. To fill, if necessary, hollows in rock with beton, or with rubble masonry.

IV. In some cases it is advisable, in order to distribute the pressure, that the rock should be covered with a layer of beton, whose thickness, in different examples, ranges from a few inches to six feet and upwards.

The intensity of the pressure on a rock foundation should at no point exceed one-eighth of the pressure which would crush the rock.

(See Article 215, p. 36.) The following are examples of the actual intensity of the pressure on some existing rock foundations:—

	Pressure in lbs. on the Square Foot.		Pressure in lbs. on the Square Inch.
Average of ordinary cases, the rock being at least as strong as the strongest red bricks, ...	20,000	...	140
Pressures at the base of St. Rollox Chimney (450 feet below the summit):—			
On a layer of strong concrete or beton, 6 feet deep,...............	6,670	...	46
On sandstone below the beton, so soft that it crumbles in the hand,.	4,000	...	28

The last example shows the pressure which is safely borne in practice by one of the weakest substances to which the name of *rock* can be applied.

The proper rule for limiting the deviation of the centre of resistance of a rock foundation from the centre of gravity of its figure is, that *there should be no tension at any point of the base.* The following is the formula for calculating the greatest value of the deviation in question which is consistent with that limitation:—*

Let A denote the area of the base;

y the distance from the centre of gravity of the figure of the base to the edge furthest from the centre of resistance;

h the total breadth of the base in the same direction;

I the moment of inertia of that figure, computed as for the cross-section of a beam relatively to a neutral axis traversing the centre of gravity at right angles to the direction of the deviation to be found. (See Article 162, pp. 252-254, and Article 179, pp. 294, 295.)

δ the deviation to be found;

then

$$\delta = \frac{\mathrm{I}}{\mathrm{A}\,y} = q\,h; \qquad (1.)$$

in which expression, q has the same meaning as in Article 179, pp. 294, 295, where a table of its values for various figures is given.

* See *Applied Mechanics*, Article 94, pp. 76, 77; Article 205, pp. 228, 229.

The only assumption involved in this equation is, that the pressure on the foundation is an *uniformly varying stress.*

237. **Theory of Earth Foundations.** (*A. M.*, 199.)—In earth whose friction is alone to be relied on for resistance to displacement by the pressure of a building, the weight of earth displaced by the foundation should not bear a less ratio to the weight of the building than that given by the following equations, in each of which

x represents the depth of the foundation;
w the weight of a cubic foot of the earth;
φ its angle of repose.

CASE I. Let the weight of the building be uniformly distributed over its base, and let p_0 be the intensity of the pressure produced by it. Then

$$\frac{w\,x}{p_0} \geq \left(\frac{1-\sin\varphi}{1+\sin\varphi}\right)^2. \quad (1.)$$

CASE II. When the weight of the building is so distributed that there is an uniformly varying pressure on the foundation, as assumed in Article 236, let p_1 be the greatest, p_2 the least, and p_0 the mean intensity of that pressure; then the two following conditions must be fulfilled:—

$$\frac{w\,x}{p_1} \geq \left(\frac{1-\sin\varphi}{1+\sin\varphi}\right)^2; \quad (2.)$$

$$\frac{w\,x}{p_2} \leq 1; \quad (3.)$$

Whence are deduced the following restrictions as to the extent of variation of the intensity of the pressure on the base, and the deviation of its centre of resistance from the centre of gravity of its figure.

$$\frac{p_1}{p_2} \leq \left(\frac{1+\sin\varphi}{1-\sin\varphi}\right)^2; \quad (4.)$$

$$\delta = q\,h \cdot \frac{p_0 - p_2}{p_0}. \quad (5.)$$

When the figure of the foundation, as is usually the case, is symmetrical about its neutral axis, we have

$$p_0 = \frac{p_1 + p_2}{2};$$

and consequently,

$$\frac{w\,x}{p_0} \geq \frac{(1-\sin\varphi)^2}{1+\sin^2\varphi} = 1 - \frac{2\sin\varphi}{1+\sin^2\varphi}; \quad (6.)$$

$$s = q\,h \cdot \frac{p_1 - p_2}{p_1 + p_2} = q\,h \cdot \frac{2 \sin \varphi}{1 + \sin^2 \varphi} \ldots\ldots\ldots\ldots (7.)$$

The following table gives some examples of the values of the functions of the angle of repose which occur in the preceding formulæ:—

φ	15°	20°	25°	30°	35°	40°	45°
$\frac{1 + \sin \varphi}{1 - \sin \varphi}$	1·700	2·039	2·464	3·000	3·690	4·599	5·826
$\frac{1 - \sin \varphi}{1 + \sin \varphi}$	0·588	0·490	0·406	0·333	0·271	0·217	0·172
$\left(\frac{1 + \sin \varphi}{1 - \sin \varphi}\right)^2$	2·890	4·159	6·070	9·000	13·619	21·152	33·94
$\left(\frac{1 - \sin \varphi}{1 + \sin \varphi}\right)^2$	0·346	0·240	0·165	0·111	0·073	0·047	0·0295
$\frac{1 + \sin^2 \varphi}{(1 - \sin \varphi)^2}$	1·945	2·579	3·535	5·000	7·310	11·076	17·47
$\frac{(1 - \sin \varphi)^2}{1 + \sin^2 \varphi}$	0·514	0·387	0·283	0·200	0·137	0·090	0·057
$\frac{2 \sin \varphi}{1 + \sin^2 \varphi}$	0·486	0·612	0·717	0·800	0·863	0·910	0·943

238. **Foundations in Firm Earth.**—When a foundation is to be made in such earth as hard clay, clean dry gravel, or clean sharp sand, —that is to say, in earth which has considerable frictional stability, and is not liable to have that stability diminished by becoming saturated with water,—it is rarely necessary to apply the principles of the preceding article; because the depth to which the foundation must be sunk, in order that the building may rest on earth below the reach of the disintegrating effects of frost and drought, is almost always greater than those principles require. In Britain that depth should be at least 3 feet for sand and 4 feet for clay. In continental regions, where the climate has greater extremes of heat and cold, a greater depth is necessary. For example, in Germany, it appears that the depths of ordinary foundations are from 4 to 5 feet, and in North America from 4 to 6 feet.

Care should be taken to divert surface-water, which may tend to run into the foundation, by means of catchwater drains, just as in other cuttings (Article 189, p. 334); and, if necessary, drains ought also to be made at the bottom of the foundation.

The greatest intensity of pressure on foundations in firm earth is

usually from 2,500 to 3,500 lbs. per square foot, or from 17 to 23 lbs. per square inch. (See p. 305.)

In fixing the "*spread*," or additional breadth given to the "footings" or foundation courses of the masonry or brickwork of ordinary walls, the usual rule is to make the breadth of the base once-and-a-half the thickness of the body of the wall in compact gravel, and twice that thickness in sand and stiff clay.

239. In **Foundations on Soft Earth** care must be taken that the depth of the foundation is not less, as compared with the pressure of the buildings, and the deviation of the centre of resistance not greater, as compared with the breadth of base, than the limits given by the formulæ of Article 237. Those objects are promoted by making the breadth of the base of the masonry as great as is practicable, so as at once to distribute its weight over a large surface, and to increase the breadth as compared with the deviation of the centre of resistance from the centre of gravity of the base.

If practicable, the ground should be well drained before the digging of the foundation is commenced, in order to increase its firmness as far as possible.

Precisely as in the case of an embankment on soft ground (Article 204, p. 343), a trench may be dug and filled with a stable material, such as sand or concrete, in order to distribute the pressure, and convey it to a sufficiently low stratum of the softer material. To find the proper depth for the trench—

Let p_1 be the greatest intensity of pressure of the intended building on its base, in lbs. per square foot. In calculating this quantity, if the trench is to be filled with sand, the area over which the weight of the building is distributed should be taken as simply equal to the area of the lowest course of foundation stones. But if the trench is to be filled with beton, the weight may be considered (as in an example given in p. 378) to be distributed over the whole area of the layer of beton, provided the edges of that layer do not project beyond the edges of the foundation stones to a distance greater than the depth of the layer of beton.

Let w be the weight in lbs. of a cubic foot of the material with which the trench is to be filled; being about 90 lbs. for sand and 130 for strong concrete or beton—

w', the weight of a cubic foot of the soft earth;

φ', its angle of repose;

also let $\frac{1 - \sin \varphi'}{1 + \sin \varphi'} = k'$; (for values of k' and of k'^2, see p. 380);

and the required depth of the trench $= x'$; then

$$x' = \frac{p_1 k'^2}{w' - w k'^2} \text{ ....(1.)}$$

The material for filling the trench should be laid and rammed in layers of about a foot deep. If concrete is used, the effect of ramming may be produced by throwing it down from a scaffolding at least ten feet high.

If the trench is filled with sand, the building may be founded on the layer of sand as on a natural sand foundation.

If the trench is filled with concrete, the building is to be built on the upper surface of that layer as soon as it is *set*, care being taken that the intensity of the pressure on the concrete does not anywhere exceed one-eighth part of its resistance to crushing; that is to say, about

$$\frac{53,300}{8} = 6,660 \text{ lbs on the square foot, or}$$

$$\frac{370}{8} = 46 \text{ lbs. on the square inch.}$$

In buildings which contain a number of openings, such as arches, windows, doorways, &c., the distribution of the load on the foundation over an increased area may be effected by means of inverted arches under the openings, provided those arches are very accurately built.

The more difficult class of foundations in soft ground, which require the use of timber or iron to make them safe, will be treated of in a later chapter.

When, by trial-pits and borings, it is shown that *a soft stratum underlies a firm one*, equation 1 should be applied in order to determine whether the depth (x') of the firm stratum is sufficient to make the foundation safe. When the firm stratum consists of sound rock, the intensity p_1 of the pressure on the soft stratum due to the weight of the building may be computed according to the same rule as for a layer of concrete. The results of this method will err on the safe side.

SECTION V.—*Construction of Stone-Masonry.*

240. **General Principles.**—The following principles are to be observed in the building of all classes of stone-masonry.

I. To build the masonry, as far as possible, in a series of courses, perpendicular, or as nearly perpendicular as possible, to the direction of the pressure which they have to bear; and to avoid all long continuous joints parallel to that pressure by "breaking joint."

II. To use the largest stones for the foundation course.

III. To lay all stones which consist of layers or "beds" in such a manner that the principal pressure which they have to bear

shall act in a direction perpendicular, or as nearly perpendicular as possible, to the direction of the layers. This is called "*laying the stone on its natural bed*," and is of primary importance to strength and durability, as has been already explained in various Articles.

IV. To moisten the surface of dry and porous stones before bedding them, in order that the mortar may not be dried too fast, and reduced to powder by the stone absorbing its moisture. (Article 229, p. 372.)

V. To fill every part of every joint, and all spaces between the stones, with mortar; taking care at the same time that such spaces shall be as small as possible.

241. **Masonry Classed.**—The *face* of a stone is its outer surface which is exposed to view. Its *beds* are the surfaces parallel to the layers. Its *sides* are the surfaces which bound it in a direction transverse both to the face and to the beds. The term *bed* is also applied to the joints between or parallel to the courses, through which the principal pressures act; these joints are also called *bed-joints;* the *side-joints,* or joints transverse both to the beds and the face, are often called simply *joints.*

The classification of masonry for engineering purposes is based almost entirely on the size and figure of the stones, and on the manner in which the joints, whether bed-joints or side-joints, are formed and executed, the appearance of the face being a matter of secondary importance.

The principal tools employed in the dressing of stone are, the scabbling hammer, whose head is pointed at one end like a pick, and axe-formed at the other, and various chisels, of which one is pointed at the end, and the others flat, and of breadths ranging from one to three inches, or thereabouts.

The scabbling hammer produces a rough approximation to a plane surface; the point gives a closer approximation, producing a surface covered with a number of small parallel ridges and furrows; the "inch-tool" and other flat-ended chisels cut away the ridges left by the point, producing still greater smoothness. Stone thus dressed is said to be "droved."

There are an indefinite number of different qualities of masonry, from "perpend ashlar," in which every stone is hewn to a regular figure and exactly fitted to the adjoining stones, to common rubble, in which the stones are built nearly as they come from the quarry, great irregularities of figure alone being reduced by means of the hammer.

For engineering purposes, masonry may be classed generally under four principal kinds, viz.:—Ashlar — Block-in-course — Coursed Rubble—and Common Rubble—and the combinations of those four kinds.

242. **Ashlar Masonry**, or hewn stone, consists of blocks cut to regular figures, generally rectangular, and built in courses of an uniform depth, which is seldom less than a foot.

In order that the stones may not be liable to be broken across, no stone of a soft material, such as the weaker kinds of sandstone and granular limestone, should have a length greater than 3 times its depth; in harder materials, the length may be 4 or 5 times the depth. The breadth, in soft materials, may range from 1½ times to double the depth; in hard materials, it may be 3 times the depth.

The bed-joints and side-joints are dressed to plane surfaces (and in some exceptional cases to be afterwards specified to curved surfaces). In the case of plane joints this is done by making an accurately plane chisel-draught all round the edges of the surface to be shaped, and if the stone is large, some additional transverse chisel-draughts in the same plane, and dressing the remainder of the surface by the point down to the plane of the chisel-draught, which serves as a guide. The accuracy with which this is done is of special importance in the case of bed-joints; for if any part of the surface projects beyond the plane of the chisel-draught, that projecting part will have to bear an undue share of the pressure, which will be concentrated upon it; and the joint, which will gape at the edges, constituting what is called an *open joint*, will be wanting in stability. On the other hand, if the surface of the bed is concave, having been dressed down below the plane of the chisel-draughts, the pressure is concentrated on the edges of the stone, to the risk of splintering them off. Such joints are said to be *flushed*. They are more difficult of detection after the masonry has been built than open joints, and are often executed by design, in order to give a neat appearance to the face of the building; and therefore their occurrence must be guarded against by careful inspection of the progress of the stone-cutting.

When the stone has been dressed so that all the small ridges on its surface are in one plane with the chisel-draughts, the pressure is distributed with a near approach to uniformity; for the mortar serves to transmit it to the furrows between the ridges.

Great smoothness is not desirable in the joints of ashlar masonry intended for strength and stability; for a moderate degree of roughness adds at once to the resistance to displacement by sliding, and to the adhesion of the mortar.

Each stone should first be fitted into its place dry, in order that any inaccuracy of figure may be discovered and corrected by the stone-cutter, before it is finally laid in mortar, and settled in its bed. No side-joint in any course should be directly above a side-joint in the course below; but the stones should overlap or *break joint* to an extent of from once to once-and-a-half the depth of a

course. This is called the *bond* of the masonry: its effect is to make each stone be supported by at least two stones of the course below, and assist in supporting at least two stones of the course above; and its objects are twofold: *first*, to distribute the pressure; so that inequalities of load on the upper part of the structure, or of resistance at the foundation, may be transmitted to and spread over an increasing area of bed in proceeding downwards or upwards, as the case may be; and *secondly*, to tie the building together, or give it a sort of tenacity, both lengthwise and from face to back, by means of the friction of the stones where they overlap.

A stone which lies with its greatest length parallel to the face of the building is called a *stretcher*. A stone which lies with its greatest length perpendicular to the face of the building is called a *header*. Stretchers tie the building together lengthwise, headers crosswise. The strongest bond in ashlar masonry is that in which each course at the face of the building contains a header and a stretcher alternately, the outer end of each header resting on the middle of a stretcher of the course below; so that rather more than *one-third* of the area of the face consists of ends of headers. This proportion may be deviated from when circumstances require it; but in every case it is advisable that the ends of headers should not form less than *one-fourth* of the whole area of the face of the building.

Quoins, or corner-stones, which should be of large size and chosen with special care, are at once headers and stretchers; each quoin being a header relatively to one of the two faces of the building which it connects, and a stretcher relatively to the other.

The *thickness of mortar* in the joints of well-executed ashlar masonry should be about an eighth of an inch. The *volume of mortar* required in all is about one-eighth part of the volume of the stone.

Ashlar masonry is used in engineering chiefly for the piers, abutments, arches, and parapets of bridges, for hydraulic works to be afterwards specified, for facing, quoins, string courses, and coping to inferior kinds of masonry, and to brickwork, and in general, for works in which great strength and stability are required.

A rougher kind of ashlar masonry is built with stones of the sizes and figures already mentioned, but scabbled or dressed with the hammer. It may be considered as intermediate between ashlar and block-in-course.

It what manner soever the faces of ashlar stones are dressed, or even should they be "quarry-faced," there ought to be a chisel-draught round the edges of the face, forming sharp and straight edges with the chisel-draught of the beds and joints, in order that the stone may be accurately set.

243. **Block-in-Course Masonry** differs from hammer-dressed ashlar chiefly in being built of smaller stones. The usual depth of the courses is from 7 to 9 inches. The same rules apply to breaking joint, and to the proportions which the lengths and breadths of the stones should bear to their depths, as in ashlar, and as in ashlar also, *at least one-fourth* of the face of the building should consist of headers, whose length should be from 3 to 5 times the depth of a course.

Block-in course masonry is used for spandrils and wing-walls of bridges, the facing of retaining walls, and similar purposes.

244. In **Coursed Rubble Masonry** the building consists of a series of horizontal courses, seldom exceeding one foot in depth, each of which is correctly levelled before another is built upon it; but the side-joints are not necessarily vertical. *One-fourth part* at least of the face in each course should consist of bond-stones or headers; each header to be of the entire depth of the course, of a breadth ranging from 1½ times to double that depth, and of a length extending into the building to from 3 to 5 times that depth, as in ashlar. Those headers should be roughly squared with the hammer, and their beds hammer dressed to approximate planes; and care should be taken not to place the headers of successive courses above each other; for that arrangement would cause a deficiency of bond in the intermediate parts of the course. Between the headers, each course is to be built of smaller stones, of which there may be one, two, or more, in the depth of the course. These are sometimes roughly squared, so as to have vertical side-joints; sometimes the stones are taken as they come, so that the side-joints are irregular; but no side joint should form an angle with a bed-joint sharper than 60°. Care should be taken, not only that each stone shall rest on its natural bed, but that the sides parallel to that natural bed shall be the largest, so that the stone may lie flat, and not be set on edge or on end. Howsoever small and irregular the stones may be, care should be taken to make the courses break joint. Hollows between the larger stones should be carefully filled with smaller stones, completely imbedded in mortar.

Coursed rubble masonry requires great care in the inspection of its progress, to see that the preceding rules are observed; and especially, that the interior of the wall contains neither empty hollows, nor spaces filled wholly with mortar or with rubbish where pieces of stone ought to be inserted, and that each stone is laid flat, and on its natural bed. Care must be taken that the headers or bond-stones are really what they profess to be, and not thin stones set on edge at the face of the wall.

A cubic yard of rubble masonry requires, in order to allow

for waste, about 1⅛ cubic yard of stones, and ⅕ cubic yard of mortar.

The resistance of good coursed rubble masonry to crushing is about four-tenths of that of single blocks of the stone that it is built with.

Coursed rubble is used for retaining walls and wing-walls that require less strength than those built of block in-course or ashlar, for the backing of pieces of masonry that are faced with ashlar or block-in-course, for fence-walls, and for various other purposes.

Rubble is often built in "random courses;" that is to say, each course rests on a plane bed, but is not necessarily of the same depth or at the same level throughout, so that the beds occasionally rise or fall by steps.

245. **Common Rubble Masonry** differs from coursed rubble in not being built in courses; but in other respects the same rules are to be observed. The resistance of common rubble to crushing is not much greater than that of the mortar which it contains; it is therefore not to be used where strength is required, unless built with strong hydraulic mortar. Its chief use in engineering is for fence walls.

246. **Ashlar and Block-in-Course backed with Rubble.**—In this sort of masonry the stones of the ashlar or block-in-course face should have their beds and joints accurately squared and dressed with the hammer, or the point, as the case may be (see Articles 242, 243, pp. 384 to 386), for a breadth of from once to twice (or on an average, once and a-half) the depth of the course, inwards from the face; but the backs of these stones may be rough. The proportion and length of the headers should be the same as in ashlar, and the "tails" of those headers, or parts which extend into the rubble backing, may be left rough at the back and sides; but their upper and lower beds should be hammer-dressed to the general planes of the beds of the course. These tails may taper slightly in breadth, but should not taper in depth.

The rubble backing, built as described in Article 244, p. 386, should be carried up at the same time with the face-work, and in courses of the same depth, the bed of each course being carefully formed to the same plane with that of the ashlar or block-in-course facing.

In estimating the labour or cost of building such masonry as is here described, the area of the face, multiplied by the distance inwards to which the dressing of the joints is carried, may be taken as ashlar or block-in-course, as the case may be, and the remainder as rubble.

These combinations of masonry are the most generally useful in engineering works; and they are especially suitable in a mechanical

point of view where the pressure is concentrated towards the face of the building, as in retaining walls.

For the abutments of bridges they are *not* mechanically suitable, because the pressure is concentrated towards the back; but if in any bridge coursed rubble is strong enough to resist the pressure at the back of the abutments, it may be used for that purpose, and faced with block-in-course, or ashlar, for the sake of appearance, and of protection from the weather.

Coursed rubble masonry is often used in combination with ashlar quoins, to which the remarks in Article 242, p. 385, are applicable.

247. **String Courses and Copes.**—A string course is a course of large stones slightly projecting beyond the face of a building, and dressed and built like ashlar or block-in-course, as the case may be. Setting aside its architectural appearance, its mechanical use is to support some load and distribute it upon the masonry below it. For example, when a coursed rubble or block-in-course wing-wall or spandril of a bridge has to support an ashlar parapet, a string course must first be placed on the wall, to give a steady base for the parapet, and to distribute its weight over the smaller stones below.

The *Cope* of a wall consists of large and heavy stones, slightly projecting over it at both sides, accurately bedded on the wall, and jointed to each other with hydraulic mortar, or with cement. Its use is to shelter the mortar in the interior of the wall from the weather, and to protect, by its weight, the smaller stones below it from being knocked off or picked out. Cope-stones should be so shaped that water may rapidly run off from them.

Rough rubble coping forms an exception to the general rule that laminated stones should be laid with their layers parallel to the beds of the courses. In this case the stones are very often set on edge, with their layers vertical, and perpendicular to the length of the wall, so that the edges of the layers alone are exposed to the air, at the top, as well as at the sides of the cope.

Additional stability is given to a cope by so connecting the cope-stones together that it is impossible to lift one of them, without, at the same time, lifting the ends of the two next it. This is done either by means of iron cramps inserted into holes in the stones, and fixed there with lead, or better still, by means of *dowels* of some very hard and strong stone, such as greenstone or granite. These are small prismatic or cylindrical blocks, each of which fits into a pair of opposite holes in the contiguous ends of a pair of cope-stones, where it is fixed with cement or hydraulic mortar.

Cast iron and wrought iron dowels are also used, but they are inferior in durability to those of hard stone, though superior in strength. Copper dowels are strong and durable, but expensive.

Cramps or dowels may be used in string courses, or in any part of a piece of masonry.

Fence-walls are sometimes coped with sods, or with clay-puddle. (Article 206, p. 344.)

248. **Pointing** a piece of masonry consists in scraping the mortar from the outer edges of the joints, at the face of the building, as far as the point of the trowel will reach, and filling the groove so made with mixed cement, or with hydraulic mortar, to keep out moisture. As to mixed cement, see Article 231, p. 374.

In sea-walls exposed to hard blows from the waves, cement put into the joints by ordinary pointing is apt to jump out in pieces; and it is best to *lay* the stones in cement for two or three inches inwards from the face of the wall.

249. **Dry Stone Walls** should be built according to the principles already laid down for rubble masonry in Articles 244, 245, pp. 386, 387, with the single exception that the mortar is to be omitted. It is often advisable to make the cope of a dry stone wall waterproof, in order that water may not lodge in the joints of the wall and force the stones from their places by its expansion in freezing. In such cases the cope may be made of stones set on edge, and jointed with mortar; or of bituminous concrete (Article 234, p. 376), or if great cheapness be desired, of clay puddle. (Article 206, p. 344.)

If a dry stone wall is intended to be permanent, rounded boulders should not be used in their natural condition to build it, but should first be broken into flat and angular pieces.

Dry stone building is employed for fence-walls, and sometimes for a backing to retaining walls, in order at once to diminish the pressure of earth against them, and to drain away water by letting it escape between the crevices of the stones.

It is also used in retaining walls of small height, and in facing earthen slopes exposed to the action of water (Article 196, p. 339; and Article 205, p. 344); and in the latter case the beds of the courses are laid perpendicular to the direction of the steepest slope.

250. **Labour of Stone-Masonry.**—The following information as to the labour required to execute different kinds of work connected with stone-masonry is given chiefly on the authority of Gauthey:—

RUBBLE STONE, one cubic yard.	Day's Work of a Man
Loading barrows with stone, ...	0·06
Wheeling one relay = about 100 feet on a level,	0·045
(As in earthwork, each foot of ascent is equivalent to *six feet* of additional distance.)	
Unloading barrows, ...	0·03

As to mixing mortar, see Article 229, p. 373; and as to the proportions of mortar and stone, see Articles 242 to 245, pp. 385 to 387.

Kinds of Building.	Day's Work of a Man per Cubic Yard. Breaking Stone.	Stone-Cutting.	Building.	Labourers' Work.
Dry stone,..........	0·64	—	1·00	0·50
Coursed rubble,.................	0·64	—	0·90	0·90
Block-in-course,..	0·90	1·5	0·90	0·90
Block-in-course arching,...	0·90	2·25	0·90	0·90
Ashlar (soft sandstone), from	1·80	2·50	1·00	1·00
Ashlar (soft sandstone), to ...	2·50	6·00	2·00	2·00

Facing ashlar, per square foot (soft sandstone)—

Stroked with the point, 0·05; Droved, 0·07; Polished, 0·10.

Labour of breaking and stone-cutting for harder stones—

Hard sandstone = soft sandstone × 2;
Hard limestone, marble, granite = soft sandstone × from 3 to 4.

$$\text{Curved facing} = \text{flat} \times \left(1 + \frac{2\frac{1}{2}}{\text{radius in feet}}\right).$$

Taking down old masonry, 0·5 day's work of a man per cubic yard.

251. **Mechanism for moving large Stones.**—There are various ways of laying hold of stones that are too heavy to be moved by hand, the most usual being the following:—

I. By nippers or tongs, the claws of which enter a pair of holes in the sides of the stone. Those holes should be situated in a horizontal line passing through or a little above the centre of gravity of the stone.

II. By a single iron plug, very slightly tapered, and driven tightly with the hammer into a vertical cylindrical hole in the top of the stone, directly above its centre of gravity. At the upper end of the plug is an eye, to which the chain for lifting the stone is hooked. After the stone has been laid in its place, a few sharp taps given sideways with the hammer loosen the plug. This method answers best with the hardest stones, such as granite.

III. By a pair of iron plugs, inserted into two holes in the top of the stone, which converge towards each other at a right angle, being inclined in opposite directions at angles of 45°. The eyes at the upper ends of the plugs are attached to a pair of chains, which, when the stone hangs by them, are at right angles to their respective plugs, and meet each other at a right angle, where they are attached to the lower end of one main chain. The two plug-

holes should be in a vertical plane traversing the centre of gravity of the stone, and equally distant from it. The tension on each of the branch chains is

$$= \text{weight of the stone} \times \cdot 707,$$

IV. By the *Lewis*, a truncated iron wedge or dovetail with the larger end downwards, made of three pieces, which can be put into or taken out of a similarly shaped hole in the top of the stone one by one, but not together. The lewis-hole is made from 2 inches to 10 inches deep, according to the weight of the stone.

The most generally useful machine for lifting and shifting large stones in ordinary buildings is the moveable jib-crane. In large buildings a travelling crab or winch is used, running on a travelling platform; that is to say, a framework of timber and iron is erected, consisting of two parallel lines of posts with sufficient diagonal bracing, supporting a pair of parallel beams, which extend along the whole length of the intended building, and include its greatest breadth between them; each of those longitudinal beams carries an iron rail; upon the pair of longitudinal rails so carried run the wheels supporting the travelling platform, which spans over the whole breadth of the building, and is made sufficiently strong and stiff by tubular iron beams or otherwise; it carries a pair of transverse rails, upon which runs a four-wheeled truck, carrying the crab or winding machine, which can thus be moved to any part of the building. The whole apparatus may be worked by a steam-engine.

252. **Instruments used in Building.**—In Article 65, p. 111, and Article 68, p. 113, it has been already explained how the situations and levels of those leading points upon which the situations and levels of all other points in a piece of masonry depend, are to be set out by the engineer.

The principal instruments used during the progress of the building are the cord, for setting out long straight lines, such as the edges of the bed-joints; the straight-edge, for shorter straight lines and for plane surfaces; the square and the bevel, for right and oblique angles; the plumb-rule, for vertical or nearly vertical lines; the level, for horizontal lines and planes, which may be like an inverted ⊥, with a plummet to set the stem vertical, or what is better, a spirit-level.

When the face of a wall is to be vertical, it can be set out, and its accuracy tested, by a plumb-rule, being a flat, straight-edged piece of board, with a line marked on it parallel to one of its edges, which line is set truly vertical by a plummet.

When the face of the wall is to have "*a straight batter*,"—that is, to be inclined at an uniform angle to the vertical, the rule to be

used is still straight-edged, but the edge is inclined to the plumb-line at the proper angle of batter. The batter of a wall is usually described by stating the extent of deviation from the vertical in a given height; for example, "one in twelve," or "one inch in a foot."

When the vertical section of the face of a wall is to be curved, it is said to have a "curved batter," and it must be set out by means of a "face-mould,"—that is to say, a narrow, flat board, having one of its edges of the intended figure of the face of the wall, and having a straight line marked upon it, which is set truly vertical by means of a plummet. Great care should be bestowed on preparing the face-moulds of important pieces of masonry; in some cases, which will be exemplified farther on, every course of stones ought to be marked on the edge of the mould.

Large face-moulds are sometimes made of several pieces of timber framed together.

When the beds of the courses are to be plane and level, they can be set correctly by the level and common straight-edge. When they are to be planes having a given slope, a rule must be employed having two straight edges inclined to each other at such an angle that, when one edge is set horizontal by the spirit-level, the other has the proper inclination. If the beds of the courses are to be perpendicular to a straight or curved battering face, their position can be set out and tested by the square.

Curved beds, such as are employed for some special purposes, require the use of suitably curved "*bed-moulds*."

In all cases in which economy of time and money has to be studied, the engineer should, as far as practicable, avoid curved figures in masonry; for not only are they more tedious and expensive to set out and to build than straight and plane figures, but it is more difficult to test the accuracy with which they have been executed. A single glance will detect the smallest appreciable inaccuracy in a wall with a straight batter, while the same process in the case of a wall with a curved batter would require either a long series of measurements, or the application of a cumbrous face-mould to various parts of the wall; and this becomes a matter of serious importance in large structures, where errors in form may affect the strength and stability.

253. **Mensuration of Masonry.**—For engineering purposes, quantities of the rougher kinds of masonry are stated in cubic yards, and of the finer, in cubic feet.

But there are also special units of measure for masonry, such as the following:—

A *rood* of masonry means, when applied to surface, 36 square yards, and when applied to volume, 36 square yards of a wall of a specified thickness, such as 2 feet. In estimating a building

according to this system, the superficial measure of the face is taken in roods of 36 square yards, in order to estimate the cost of the face-work; and then the area in superficial roods of the face of each portion of the building is multiplied by the ratio which its thickness bears to 2 feet, so as to compute the cubic contents in solid roods of 36 square yards in area and 2 yards thick in order to estimate the cost of the masonry exclusive of the face. This method is better suited to architectural than to engineering purposes.

Section VI.—*Construction of Brickwork.*

254. **General Principles.**—The following principles are to be observed in building with bricks:—

I. To reject all misshapen and unsound bricks. (See Article 220, p. 366.)

II. To place the beds of the courses perpendicular, or as nearly perpendicular as possible, to the direction of the pressure which they have to bear; and to make the bricks in each course break joint with those of the courses above and below by over-lapping to the extent of from one quarter to one half of the length of a brick.

III. To cleanse the surface of each brick, and to wet it thoroughly before laying it, in order that it may not absorb the moisture of the mortar too rapidly.

IV. To fill every joint thoroughly with mortar, taking care at the same time that the thickness of mortar shall not exceed about a quarter of an inch.

In order to prevent the use of too great a thickness of mortar, it is usual in specifications to prescribe a certain depth which a certain number of courses of brickwork shall not exceed. For example, if the bricks are $2\frac{3}{4}$ inches deep, it may be specified that four courses of bricks, when built, shall not measure more than one foot in depth; a condition which implies that the average thickness of mortar in the joints shall be $\frac{1}{4}$ inch.

V. To use no "bats," or pieces of bricks, except when absolutely necessary, in order to make a "closure,"—that is, to finish the end or corner of a wall, or the side of an opening; and even then, to use no piece less than half a brick.

In stating the length and breadth of masses of brickwork, it is usual to employ the length of a brick as an unit of measure. For example, if bricks are used which build to 9 inches in length,

$\frac{1}{2}$	brick	means	$4\frac{1}{2}$ inches.
1	"	"	9 inches.
$1\frac{1}{2}$	"	"	1 foot $1\frac{1}{2}$ inch.
2	"	"	1 foot 6 inches.

And so on.

The *volume of mortar* required for good brickwork is about *one-fifth* of the volume of the bricks.

255. **Bond in Brickwork.**—The bricks used in a given building

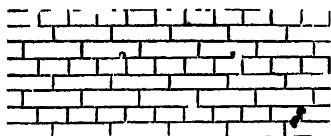


Fig. 166.

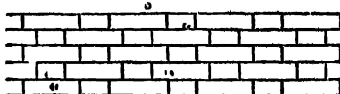


Fig. 167.

being of uniform or nearly uniform size and figure, are to be built according to an uniform system, which is called the *bond* of the brickwork.

As in ashlar masonry, so in brickwork, a *header* is a brick whose length lies perpendicular to the face of the wall; a *stretcher*, one whose length lies parallel to the face of the wall. As the length of a brick is almost exactly double of its breadth, one stretcher occupies the same area on the face of the wall with two headers.

I. *English Bond*, which is considered the strongest and most stable arrangement, consists in laying entire courses of headers and of stretchers periodically, as in fig. 166. Sometimes the courses of headers and stretchers occur alternately; sometimes there is only one course of headers for every two, three, or four courses of stretchers. The stretchers tie the wall together lengthwise, the headers crosswise. The proportionate numbers of the courses of headers and stretchers should depend on the relative importance of transverse and longitudinal tenacity. (*A. M.*, 202.) The proportion shown in fig. 166, of one course of headers to two of stretchers, is that which gives equal tenacity to the wall lengthwise and crosswise, and which therefore may be considered the best in ordinary cases.

In a factory chimney, the longitudinal tenacity, which resists any force tending to split the chimney, is of more importance than the transverse tenacity; therefore, in these buildings, it is advisable to have a greater proportion of stretchers, such as three or four courses of stretchers to one course of headers.

In building brickwork in English Bond, it is to be borne in mind that there are twice as many vertical or side-joints in a course of headers as there are in a course of stretchers; and, therefore, that unless great care is taken in laying the headers to make these joints very thin, two headers will occupy a little more length than one stretcher, and the correct breaking of the joints, to the extent of exactly a quarter of a brick, will be lost. This is often the case in carelessly built brickwork, in which at intervals vertical joints are seen nearly or exactly above each other in successive courses.

II. In *Flemish Bond* (fig. 167) a header and a stretcher are laid alternately in each course, and so placed that the outer end of each header lies on the middle of a stretcher in the course below. The number of vertical joints in each course is the same, so that there is no risk of the correct breaking of the joints by a quarter of a brick being lost; and the wall presents a neater appearance than one built in English Bond. English Bond, however, when correctly built, is considered to be stronger and more stable than Flemish Bond.

256. **Hoop Iron Bond.**—Pieces of hoop iron are sometimes laid flat in the bed-joints of brickwork, to increase its longitudinal tenacity. They should break joint with each other; and the ends of each piece of hoop iron should be bent down at right angles for the length of two inches or thereabouts, and inserted into vertical joints of the course of bricks on which the hoop iron lies.

The total sectional area of the hoop iron needs not exceed about 1-300th of that of the brickwork.

257. **Pointing Joints.**—(See Article 248, p. 388.)

258. The **Foundation Courses** of a piece of brickwork usually spread downwards by steps of a quarter of a brick at the face and back, until a sufficient breadth is gained to support the weight of the building, according to the principles already explained in Section IV. of this Chapter, pp. 377 to 381.

259. **String Courses and Copes.** Brick string courses ought to consist entirely of headers, and so also ought copes built with ordinary bricks. Coping for brick walls is sometimes made with large bricks moulded expressly for that purpose. Stone string courses and coping are frequently used along with brick building, especially where strength and stability are required.

260. In **Brickwork with Stone Quoins** special care must be taken that the layer of mortar in each bed-joint of the brickwork is as thin as possible; for as the bed-joints of the brickwork are three or four times as numerous as those of the stone quoins, any superfluous thickness of the former will cause the brickwork to settle more than the stone quoins, the effect of which will be to disfigure, crack, and perhaps destroy the building.

261. **Labour of Brickwork.**—The following data are given on the authority of Gauthey:—

Day's Work of a Man per Cubic Yard.

	Bricklayer.	Labourer.	Erecting Scaffolding.
Ordinary Bricklaying,......	0·6	0·6	0·2
Brick Arching,..............	0·9	0·9	various.

In the case of arching, the labour of erecting scaffolding includes

that of putting up "centres," or timber frames for the arches to rest on while in progress. These structures will be considered further on.

262. **Mensuration of Brickwork.**—For engineering purposes in Britain, quantities of brickwork are usually computed and stated in cubic yards; but there are also special modes of stating them, such as the following:—

Each piece of brickwork has its thickness stated in bricks and half bricks; the area of its face is calculated: from that area is computed the area of a wall of the standard thickness of *one brick and a-half*, and of the same cubic contents, and the reduced area so obtained is stated in *rods* of $30\frac{1}{4}$ square yards, and in square yards.

A *rod* of brickwork, of a brick and a-half thick, if each brick be 9 inches long, is equal to $11\frac{1}{3}$ cubic yards very nearly.

SECTION VII.—*Of Buttresses and Retaining Walls.*

263. The **Stability of Blocks of Masonry and Brickwork in General** (*A.M.*, 211) depends on the conditions already stated in Article 139, p. 220—viz., that of *stability of position*, which requires that the structure shall not give way by overturning, and that of *stability of friction*, which requires that it shall not give way by the sliding of one course upon another; and those two conditions ought to be fulfilled *at the bed-joint of each course.*

The following are the most convenient ways of expressing these conditions by means of formulæ suitable for calculation:—

I. *Stability of Position* is insured when the moment of the force tending to overturn the mass above a given bed-joint does not exceed the moment of stability of the mass of masonry above that bed-joint.

To express the *moment of stability at a given bed-joint* symbolically, it is necessary, in the first place, to determine the greatest distance to which the "*centre of pressure*" or "*of resistance*" at that bed-joint may deviate from the middle of the bed, without endangering the stability of the structure.

Let q denote the greatest safe ratio of the deviation to the thickness of the masonry at the given bed-joint.

In *flying buttresses*, and *piers and abutments* of arches and of frames, it is in general advisable to limit q according to the rule already given for rock foundations, Article 236, p. 378—viz., *that there shall be no tension at any point of the bed*, the pressure being supposed to be an uniformly varying stress. For various values of q, see Article 179, pp. 294, 295. The value of most common occurrence is that for solid rectangular structures—viz., $q = \frac{1}{6}$.

In *retaining walls* for sustaining the pressure of earth or of water, the following are average values of q deduced from the dimensions of actual retaining walls:—

According to the practice of British engineers, $q = \cdot 375$ nearly.

According to the practice of French engineers, $q =$ from $\cdot 3$ to $\cdot 25$.

The following is a method of determining the greatest value of q for a rectangular structure, consistent with safety from crushing of the material, based on the supposition that the intensity of the pressure diminishes at an uniform rate from the compressed edge of the bed-joint inwards, that the mortar exerts no appreciable tension, and that consequently the distance of the centre of resistance from the compressed edge is one-third of the thickness throughout which the pressure is distributed.* (See pp. 777 & 780.)

Let R be the total pressure at the given bed-joint;

b the breadth, } of the mass of masonry at that joint, in feet;
t the thickness }

f' the greatest safe pressure in lbs. on the square foot (being about one-eighth of the crushing pressure); then

$$f' = 2\,R \div \left(\frac{3}{2} - 3\,q\right) b\,t; \text{ and therefore,}$$

$$q = \frac{1}{2} - \frac{2\,R}{3\,f'\,b\,t}. \qquad \text{(1.)}$$

The value of q having been fixed, let

$r\,t$ denote the distance from the middle point of the bed to the point where the bed is cut by a vertical line let fall from the centre of gravity of the mass of masonry above it;

W, the weight of that mass; and

j, the inclination to the horizon of a line in the plane of the bed, connecting the limiting position of the centre of resistance with the point directly below the centre of gravity before mentioned.

Then the moment of stability is,

$$M = W\,(q \pm r)\,t\cos j; \qquad \text{(2.)}$$

the sign $\left\{ {+ \atop -} \right\}$ being used according as the centre of resistance, and the vertical line through the centre of gravity, lie towards $\left\{ {\text{opposite sides} \atop \text{the same side}} \right\}$ of the middle of the diameter.

The following modification of this expression is convenient in comparing structures of similar figures and different dimensions:—

Let h denote the height of the structure above the middle of the given bed-joint, b the breadth of that bed in a direction perpendicular or conjugate to the thickness t, and w the weight of an unit of volume of the material. Then

$$W = n \cdot w h b t, \quad \text{..........................(3.)}$$

where n is a *numerical factor* depending on the *figure* of the structure, and on the angles which the dimensions, h, b, t, make with each other; that is, the angles of obliquity of the co-ordinates to which the figure of the structure is referred. Introducing this value of the weight of the structure into the formula 2, we find the following value for the moment of stability:—

$$M = n (q \pm r) \cos j \cdot w \cdot h b t^2 \quad \text{..............(4.)}$$

This quantity is divided by points into three factors, viz.:—

(1.) $n (q \pm r) \cos j$, a *numerical factor*, depending on the *figure* of the structure, the *obliquities* of its co-ordinates, and the *direction* in which the applied force tends to overturn it.

(2.) w, the heaviness of the material.

(3.) $h b t^2$, a geometrical factor, depending on the dimensions of the structure.

Now the first factor is the same in all structures having figures of the same class, with co-ordinates of equal obliquity, and exposed to similarly applied external forces; that is to say, to all structures whose figures, together with the lines of action of the applied forces, are *parallel projections of each other, with co-ordinates of equal obliquity*. (See Articles 101, 140, pp. 150, 220.) Hence for any set of structures which fulfil that condition, the moments of stability are proportional to

The heaviness of the material;

The height;

The breadth;

The *square* of the thickness; that is, of the dimension of the base which is parallel to the vertical plane of the applied force.

The following is the general expression for the moment, relatively to the limiting position of the centre of resistance, of an externally applied force, tending to overturn the mass of masonry above the given bed-joint.

Let P denote the magnitude of that force;

θ the angle which its direction makes with the horizon in a direction *contrary* to that of the slope j of the bed;

x', the vertical height, and y', the horizontal distance of its point of application from the centre of resistance of the bed;

then the perpendicular distance of P from the centre of resistance is

$x' \cos \theta - y' \sin \theta$; and the required moment is given by the following formula, which also expresses the condition, that that moment shall not exceed the moment of stability of the masonry:—

$$\mathrm{P}\,(x' \cos \theta - y' \sin \theta) \not> \mathrm{M}. \qquad \text{(5.)}$$

II. *Stability of Friction* is insured when the resultant pressure makes with a normal or line perpendicular to the bed, an angle not exceeding the angle of repose of the materials.

Let ϕ denote that angle. (See Article 110, p. 172.)

The angle made by the resultant pressure with the vertical is

$$\operatorname{arc}\cdot\tan\cdot\frac{\mathrm{P}\cos\theta}{\mathrm{W}+\mathrm{P}\sin\theta};$$

and the condition of stability of friction is given by the equation,

$$\operatorname{arc}\cdot\tan\cdot\frac{\mathrm{P}\cos\theta}{\mathrm{W}+\mathrm{P}\sin\theta} - j \not> \phi. \qquad \text{(6.)}$$

This condition can always be fulfilled by properly adjusting the declivity of the bed-joint, j.

264. **Stability of a Vertical-faced Buttress with Horizontal Beds.** (*A. M.*, 213.)—Let fig. 168 represent a vertical section of a buttress, with a vertical face C D, against which a strut, rib, or piece of framework abuts at C, exerting a given force P in a given direction C A. In order that the buttress may be stable, it must fulfil the conditions of stability at each of its horizontal bed-joints. Let D E be one of those joints.

Should several pressures abut against the buttress, the force P acting in the line C A may be held to represent the resultant of all the forces which are applied above the particular joint D E under consideration.

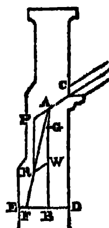


Fig. 168.

Let G be the centre of gravity of that part of the buttress which is above the joint D E, and let W denote the weight of the same part. Through G draw the vertical line A G B, cutting the direction of the lateral thrust in A, and the joint D E in B; make $\overline{\mathrm{A\,W}} = \mathrm{W}$, $\overline{\mathrm{A\,P}} = \mathrm{P}$; complete the parallelogram A P R W; then $\overline{\mathrm{A\,R}}$ will represent the resultant of all the forces which act on the part of the buttress above the joint D E, to which the resultant of the resistance at that joint must be equal and directly opposed. A R being produced, cuts D E in F, the centre of resistance of that joint, which must not fall beyond a certain prescribed limit, that the condition of

stability of position may be fulfilled. In order that the condition of stability of friction may be fulfilled, the angle A F B must not be less than the complement of the angle of repose.

In expressing this algebraically, it is to be observed that

$$\text{C D} = x'; \ \text{D F} = y'; \ j = 0;$$

and consequently that equation 5 of the preceding Article, p. 399, becomes,

$$\text{P} \left(x' \cos \theta - y' \sin \theta \right) \leqq n \left(q \pm r \right) w \, h \, b \, t^2; \ldots\ldots\ldots(1.)$$

and equation 6,

$$\frac{\text{P} \cos \theta}{n \, w \, h \, b \, t + \text{P} \sin \theta} \leqq \tan \phi. \ldots\ldots\ldots\ldots\ldots\ldots(2.)$$

By means of these fundamental equations the following problems are solved.—

I. The relation between the weight and the dimensions of the part of the buttress under consideration being given (in other words, the factor n being given), it is required to find the least thickness t at the joint D E consistent with stability of position.

In equation 1, make $y' = \left(q + \frac{1}{2} \right) t$, and put $=$ instead of $\leqq$; then

$$n \left(q + r \right) w \, h \, b \, t^2 = \text{P} \left(x' \cos \theta - \left(q + \frac{1}{2} \right) t \sin \theta. \right)$$

To simplify the form of this quadratic equation, make,

$$\frac{\text{P} \, x' \cos \theta}{n \left(q + r \right) w \, h \, b} = \text{A}, \ \frac{\left(q + \frac{1}{2} \right) \text{P} \sin \theta}{2 \, n \left(q + r \right) w \, h \, b} = \text{B};$$

then it becomes

$$t^2 = \text{A} - 2 \, \text{B} \, t,$$

the solution of which is

$$t = \sqrt{(\text{A} + \text{B}^2)} - \text{B} \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots(3.)$$

II. To find the least weight of material above the point C, consistent with stability of friction.

The greatest obliquity of pressure occurs at that joint which is immediately below the point of abutment, C. Let h_0 denote the height of material above that joint, b_0 the breadth, and t_0 the required thickness; then,

$$n \, w \, h_0 \, b_0 \, t_0 = \text{P} \left(\frac{\cos \theta}{\tan \phi} - \sin \theta \right). \ldots\ldots\ldots\ldots\ldots(4.)$$

III. *Particular Case—Rectangular Buttress.* (*A. M.*, 214.)—In a rectangular buttress, the breadth b and thickness t are constant; and if h_0 be taken to denote the height of the top of the buttress above the point C,

$$h = h_0 + x$$

will be its height above a given joint. Also, because the centre of gravity of the portion above any bed-joint is vertically above the centre of the joint, $r = 0$; and because

$$W = w\,h\,b\,t,$$

$n = 1.$

These values being substituted in equation 3, give the following results, in which x denotes the depth of the base of the wall below C.

$$A = \frac{P\,x\cos\theta}{q\,w\,(h_0 + x)\,b};\quad B = \frac{\left(q + \frac{1}{2}\right)P\sin\theta}{2\,q\,w\,(h_0 + x)\,b};\quad t = \sqrt{(A + B^2)} - B \qquad (5.)$$

As the depth x increases without limit, the thickness required for the wall approaches the following limit:—

$$t = \sqrt{\left(\frac{P\cos\theta}{q\,w\,b}\right)} \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots (6)$$

which depends on the horizontal component of the applied force alone.

Supposing this value to be adopted for the thickness of the buttress, in order that it may be stable, how deep soever the base may be below the point C, then to insure stability of friction, the height of the top above C must have the following value:—

$$h_0 = q\,t \cdot \frac{\cos(\varphi + \theta)}{\sin\varphi\cos\theta} = \frac{\cos(\varphi + \theta)}{\sin\varphi} \cdot \sqrt{\left(\frac{q\,P}{w\,b\cos\theta}\right)} \quad \ldots\ldots\ldots (7)$$

Instead of the rectangular mass $h_0\,b\,t$, there may be substituted a *pinnacle* of the same weight, and of any figure.

265. **Stability of Retaining or Revetement Walls in General.** (*A. M.*, 217.)—Figs 169 and 170 represent vertical sections of retaining walls against which banks of earth abut. In each figure a vertical layer of the masonry and of the earth is supposed to be considered, whose length is unity. D E is the base of the layer of masonry, F the centre of resistance of that base, B a point vertically below G, the centre of gravity of the weight which rests on that base, $\overline{AW}$ a line representing that weight, $\overline{AP}$ a line representing the thrust of the earth; $\overline{AR}$, the diagonal of the parallelogram A P R W, is a

line representing the resultant pressure at the base DE, and cutting that base in the centre of resistance F.

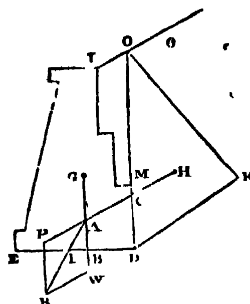


Fig 169

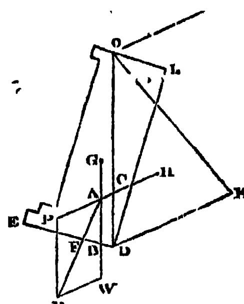


Fig 170.

In each figure, D O is a vertical plane traversing the inner edge D of the base of the wall, and cutting the plane of the surface of the bank in O. In fig 169, the whole of the wall lies in front of that vertical plane, so that the weight, represented by AW (or by W simply), which rests on the base D E, consists of the weight of the masonry *together with the weight of the mass of earth, if any* (represented by O L M), *which is vertically above that base;* and G is the common centre of gravity of the compound mass of masonry and earth, which is situated in front of the plane O D.

In fig 170, on the other hand, a part of the masonry, represented by D L O, lies *behind* the plane O D. If the prism D L O consisted of earth, its weight would be supported by the earth beneath it; therefore the earth beneath that prism exerts a pressure vertically upwards sufficient to sustain the weight of a prism of earth of a volume equal to that of the prism of masonry; therefore the weight represented by A W (or by W simply), which rests on the base D E, consists of the weight of the masonry in the vertical layer of the wall, *less* the weight of earth which would fill D L O; and G is the common centre of gravity of the masonry E D O which lies before the plane O D, and of the prism D L O, considered as having a heaviness equal to the *excess of the heaviness of masonry above that of earth.*

It has already been shown in Article 193, Division IV., p. 323 that the pressure of the earth against the vertical plane O D (which pressure is parallel to the surface of the bank, and represented by A P or by P simply), is equal to the weight of the prism of earth O DK, in which D K, parallel to the surface of the bank, is equal

to the vertical depth O D multiplied by the ratio of the conjugate pressures at a point,

$$\frac{p'}{p} = \frac{\cos\theta - \sqrt{(\cos^2\theta - \cos^2\phi)}}{\cos\theta + \sqrt{(\cos^2\theta - \cos^2\phi)}},$$

which ratio depends on the slope θ of the bank, and angle of repose ϕ; and that the resultant of that pressure traverses C, at the height

$$\overline{\text{C D}} = \frac{\overline{\text{O D}}}{3} = \frac{x}{3}$$

above D. For the sake of brevity (w' being the weight of unity of volume of the earth), let

$$w' \cos\theta \frac{p'}{p} = w_1;$$

then equation 18 of Article 183, p. 324, becomes

$$\text{P} = \frac{w_1 x^2}{2} \text{ ....(1.)}$$

This force has to be multiplied, as in Article 263, by the perpendicular distance of F from C P, to give the moment of the couple which tends to overturn the wall. Let t be the thickness $\overline{\text{D E}}$, and j the angle of inclination of D E to the horizon; then the arm of the couple in question is

$$\frac{x\cos\theta}{3} - \left(q + \frac{1}{2}\right) t \cdot \sin(\theta + j);$$

which being multiplied by the force P, and equated to the moment of stability of the weight which rests on the base D E, gives the following condition of stability of position:—

$$\text{W}(q \pm q')\, t \cdot \cos j = \frac{w_1 x^3 \cos\theta}{6} - \frac{w_1 x^2 t}{2}\left(q + \frac{1}{2}\right)\sin(\theta + j) \text{..(2.)}$$

Now suppose (as in Article 263, p. 398) that W bears a definite ratio n to the weight $w\, x\, t \cdot \cos j$ of a rectangle of masonry whose height is $\overline{\text{O D}} = x$, and its breadth the horizontal distance of E from O D, $t \cos j$; then the first side of equation 2, being the moment of stability, becomes as follows:—

$$n\,(q \pm q')\, w\, x\, t^2 \cos^2 j.$$

Divide both sides of the equation by

$$n\,(q \pm q')\, w\, x^3 \cos^2 j,$$

and for brevity's sake, let

$$\frac{w_1 \cdot \cos\theta}{6\,n\,(q \pm q')\,w\cos^2 j} = a,$$

$$\frac{w_1\left(q + \frac{1}{2}\right)\sin(\theta + j)}{4\,n\,(q \pm q')\,w\cos^2 j} = b;$$

then

$$\frac{t^2}{x^2} = a - 2\,b\,\frac{t}{x} \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (3.)$$

and consequently

$$\frac{t}{x} = \sqrt{a + b^2} - b \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (4.)$$

The inclination of the resultant A R to the vertical is given by the equation

$$\tan \angle \mathrm{W\,A\,R} = \frac{\mathrm{P}\cos\theta}{\mathrm{W} + \mathrm{P}\sin\theta} \ldots\ldots\ldots\ldots (5.)$$

When the base D E is horizontal, this should not exceed the tangent of the angle of repose When that base is inclined at the angle j, the condition of frictional stability is thus expressed:—

$$\angle \mathrm{W\,A\,R} - j \not> \varphi', \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (6.)$$

φ' being the angle of repose of the foundation of the wall.

The object of giving the base of the wall an inclined position is to diminish the obliquity of the pressure on it, and so to enable the condition of frictional stability to be fulfilled.

As to the values of q in practice, see Article 263, pp. 396, 397.

266. **Stability of Upright Rectangular Retaining Walls.** (*A. M.*, 218.)—In a vertical rectangular wall, $n = 1$, $q' = 0$, $j = 0$; so that, in equations 3 and 4 of Article 265,

$$a = \frac{w_1 \cos\theta}{6\,q\,w};\; b = w_1\left(q + \frac{1}{2}\right)\sin\theta \div 4\,q\,w \ldots\ldots\ldots (1.)$$

CASE I. When the surface of the bank is horizontal, so that $\theta = 0$, then

$$w_1 = w'\,\frac{1 - \sin\varphi}{1 + \sin\varphi};\; b = 0;$$

and the proportion of the thickness of the wall to its height is

$$\frac{t}{x} = \sqrt{a} = \sqrt{\left\{\frac{w'(1-\sin\varphi)}{6\,q\,w\,(1+\sin\varphi)}\right\}} \qquad (2.)$$
$$= \tan\left(\frac{90^\circ - \varphi}{2}\right)\sqrt{\frac{w'}{6\,q\,w}}.$$

Equation 5 of Article 265 becomes

$$\tan \angle \mathrm{W\,A\,R} = \frac{\mathrm{P}}{\mathrm{W}} = \frac{w_1 x}{2\,w\,t}$$
$$= \sqrt{\left\{\frac{3\,q\,w'(1-\sin\varphi)}{2\,w\,(1+\sin\varphi)}\right\}} \angle \tan\varphi' \qquad \ldots\ldots\ldots(3.)$$

If the material on which the wall rests is the same with that of the bank, we may assume $\varphi' = \varphi$; in which case, by squaring equation 3, and attending to the fact that

$$\tan^2\varphi = \frac{\sin^2\varphi}{1-\sin^2\varphi} = \left(\frac{\sin\varphi}{1-\sin\varphi}\right)^2 \cdot \frac{1-\sin\varphi}{1+\sin\varphi},$$

we obtain the equation

$$\frac{3\,q\,w'}{2\,w} \angle \left(\frac{\sin\varphi}{1-\sin\varphi}\right)^2 \ldots\ldots\ldots\ldots\ldots\ldots(4.)$$

Assuming that the specific gravity of the earth is four-fifths of that of the masonry, or $w' \div w = 5 \div 4$, we find that this equation is fulfilled for the ordinary value of q, $3 \div 8$, so long as φ exceeds 24°. Should equation 4 not be fulfilled for $q = 3 \div 8$, a smaller value of q must be determined by the following equation:—

$$q = \frac{2\,w}{3\,w'} \cdot \left(\frac{\sin\varphi}{1-\sin\varphi}\right)^2, \ldots\ldots\ldots\ldots\ldots\ldots\ldots (5.)$$

and introduced into equation 2 to find the ratio $t \div x$.

CASE II. When the surface of the bank slopes at the angle of repose φ, then $w_1 = w' \cos\varphi$, and

$$a = \frac{w'\cos^2\varphi}{6\,q\,w};\; b = \frac{\left(q+\frac{1}{2}\right) w'\cos\varphi\sin\varphi}{4\,q\,w} \qquad (6).$$

which values, being introduced into equation 4 of Article 265, p. 404, give the ratio $t \div x$.

267. **Stability of Battering-faced Retaining Walls.** (*A. M.*, 219.)—In fig. 171, let E Q represent the vertical face of a rect-

angular wall, suited to sustain the thrust of a given bank, and let F be the centre of resistance of the base. Make $\overline{\mathrm{Q\,N}} = 3\ \mathrm{E\,F} = 3\left(\frac{1}{2} - q\right) t$; then the centre of gravity g of the triangular prism of masonry E Q N will be vertically above the centre of resistance F; therefore, if that prism be removed, so as to reduce the cross-section of the wall to a trapezoid with a battering face E N, the position of the centre of resistance F will not be altered, and the wall will still fulfil the condition of stability of position, the thickness t being determined as for a rectangular wall. The thickness of the wall at the summit is

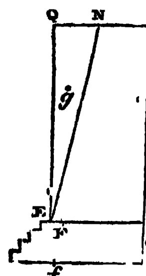


Fig. 171.

$$\left(3\,q - \frac{1}{2}\right) t.$$

The tangent of ∠ W A R (the inclination of the resultant pressure to the vertical) is increased in the ratio $\frac{1}{4} + \frac{3\,q}{2} : 1$; so that it may in some cases be necessary to make the base slope backwards, as in fig. 170.

268. **Stability of Battering Walls of Uniform Thickness.** (*A. M.*, 220.)—When a wall for supporting a *horizontal-topped* bank is of uniform thickness, and has a sloping or curved face, as in figs. 172 and 173, its moment of stability may be determined with a degree of accuracy sufficient for practical purposes, in the following manner:—

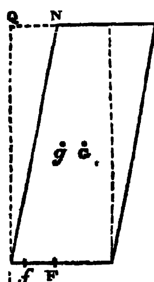


Fig. 172.

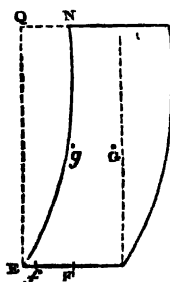


Fig. 173.

Let E Q in each figure represent the vertical face of a rectangular wall of the same height x and thickness t with the proposed wall, and let g be the centre of gravity of that rectangular wall. Then

$$\mathrm{W} \cdot q\,t = q\,w\,x\,t^2$$

will be its moment of stability per unit of length.

Divide the area E Q N included between the vertical face E Q and the face of the proposed wall, E N, by the height x. Then

$$q'\,t = \overline{g\,\mathrm{G}} = \frac{\mathrm{E\,Q\,N}}{x}, \qquad (1.)$$

will be the distance of the centre of gravity G of the sloping or curved wall from that of the rectangular wall; and the change of figure will increase the stability in the ratio $q + q' : q$; that is to say, the moment of stability will now be

$$\mathrm{W}\,(q + q')\,t = (q + q')\,w\,x\,t^2 \;\ldots\ldots \qquad \ldots (2.)$$

If E N is a straight line (fig. 172),

$$q'\,t = \overline{\mathrm{Q\,N}} \qquad .(3.)$$

If E N is a parabolic arc,

$$q'\,t = \frac{\overline{2\,\mathrm{Q\,N}}}{3}; \qquad .\,(4.)$$

a formula which is also sensibly correct when E N is an arc of a circle.

Walls with a "curved batter" are usually built as shown in fig. 174, with the bed-joints perpendicular to the face of the wall. This diminishes the obliquity of the pressure on the base.

269. **Counterforts** (*A. M.*, 222) are projections from the inner face of a retaining wall. A wall and its counterforts, if the bond of the masonry is well preserved by means of long bond-stones connecting the counterforts with the wall, are equivalent to a wall having successive divisions of its length alternately of greater and of less thickness. The moment of stability of such a wall, per

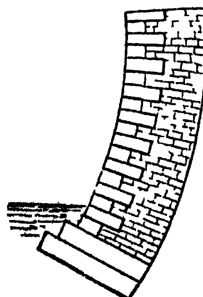


Fig. 174.

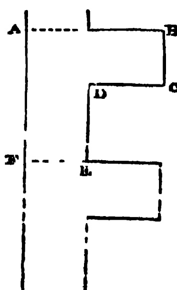


Fig 175.

unit of length, when the wall is well bonded, may be found, with sufficient accuracy for practical purposes, by adding together the moments of stability of one of the parts between two counterforts, and of one of the parts whose thickness is augmented

by the addition of a counterfort, and dividing the sum by the joint length of those two parts.

For example, let fig. 175 represent a portion of the *plan*, or horizontal section, of a vertical rectangular retaining wall whose height is h, with a row of rectangular counterforts of the same height with the wall. Let t = F E be the thickness of a part of the wall between two counterforts, and b = E D its length; let T = A B be the thickness of a counterforted part of the wall, including the counterfort, and c = B C its length.

The moment of stability of the first part is

$$q\,w\,h\,b\,t^2;$$

and that of the second part,

$$q\,w\,h\,c\,\mathrm{T}^2.$$

Adding together those moments, and dividing their sum by the total length $b + c$ = A F, the mean moment of stability per unit of length is found to be

$$q\,w\,h \cdot \frac{b\,t^2 + c\,\mathrm{T}^2}{b + c} \quad \text{..........................(1.)}$$

This is the same with the moment of stability per unit of length of a wall of the uniform thickness,

$$\sqrt{\left\{\frac{b\,t^2 + c\,\mathrm{T}^2}{b + c}\right\}} \quad \text{.......................(2.)}$$

which may be called the *equivalent uniform wall.*

The quantity of masonry in the counterforted wall is to the quantity in the equivalent uniform wall in the ratio

$$b\,t + c\,\mathrm{T} : (b + c)\,t_1,$$

which is always less than unity, so that there is a saving of masonry (though in general but a small one) by the use of counterforts.

270. **Surcharged Retaining Wall.**—This term is applied to a wall for supporting a bank of earth which rises from the top of the wall at the natural slope for a certain height, called the ***height of the surcharge***, beyond which it is horizontal. The thrust of such a bank is intermediate in amount and in direction between that of a horizontal-topped bank and that of a bank with an indefinitely long natural slope.

The following formula serves to compute approximately the thickness of an upright rectangular retaining wall for supporting such a bank:—

x, as before, denote the height of the wall,
c the height of the surcharge,
t the thickness required to sustain a horizontal bank, as computed by equation 2 of Article 266, p. 405,
t' the thickness required to sustain a bank with an indefinitely long natural slope, as computed by equations 6 of Article 266, p. 405, and 4 of Article 265, p. 404,
t'' the thickness required for the surcharged wall, then

$$t'' = \frac{x t + 2 c t'}{x + 2 c}, \text{ nearly.} \quad \ldots \quad (1.)$$

When the foot of the slope of the bank rests on the top of the wall, nearly above the centre of resistance of the base, the following formula may be used:—

$$\frac{t''}{x} = \cos \phi . \sqrt{\left(\frac{w'}{6 q w}\right) \cdot \left(\frac{x}{1 + \sin \phi} + 2 c\right) \div (x + 2 c)} \ldots \ (2.)$$

271. **Construction of Retaining Walls.**—The foundation courses of retaining walls have their width increased beyond the thickness of the wall by a series of steps in front, as shown in figs. 171 and 174. The objects of this are, at once to distribute the pressure over a greater area than that of any bed-joint in the body of the wall, and to diffuse that pressure more equally, by bringing the centre of resistance nearer to the middle of the base than it is in the body of the wall, according to the principles already explained in Section IV. of this Chapter, pages 377 to 382.

The body of the wall may be either entirely of brick, or of ashlar backed with brick or with rubble, or of block-in-course backed with rubble, or of coursed rubble, built with mortar or built dry. As the pressure at each bed-joint is concentrated towards the face of the wall, those combinations of masonry in which the larger and more regular stones form the face, and sustain the greater part of the pressure, and are backed with an inferior kind of masonry, whose use is chiefly to give stability by its weight, are well suited for retaining walls (see Article 246, p. 387), special care being taken that the back and face are well tied together by long headers, and that the beds of the facing stones extend into the wall to a distance of about as far inwards from the centre of pressure at the base of the wall as that centre of pressure lies inwards from the face.

Along the base and in front of a retaining wall there should run a drain, like that at the foot of the slope of a cutting. (See Article 193, p. 335.) In order to let water escape from behind the wall, it has small upright oblong openings through it, called "weeping-holes," which are usually two or three inches broad, and of the depth of a course of masonry, and are distributed at regular distances, an

ordinary proportion being one weeping-hole to every 4 square yards of face of wall.

The back of a retaining wall should be rough, in order to resist any tendency of the earth to slide upon it. This object is promoted by building the back in steps, as exemplified in fig. 169.

In retaining walls of great size, both back and face may be built of block-in-course, or the face of ashlar and the back of block-in-course, the "heart" of the wall being of coursed rubble, or of beton or strong concrete laid in regular courses of the same depth with those of the face and back.

When the material at the back of the wall is clean sand or gravel, so that water can pass through it readily, and escape by the weeping-holes, it is only necessary to ram it in layers, as already described in Articles 198 and 200, pp. 341, 342. But if the material is retentive of water, like clay, a *vertical layer* of stones or coarse gravel, at least a foot thick, or a dry stone rubble wall, must be placed at the back of the retaining wall, between the earth and the masonry, to act as a drain.

A catchwater drain behind a retaining wall is often useful. It may either have an independent outfall, or may discharge its water through pipes into the drain in front of the base of the wall.

When the material at the back of the wall is of a loamy description, and liable to be reduced to quicksand or mud by saturation with water, and there are no means of preventing such saturation by efficient drainage, one way of making provision to resist the additional pressure which may arise from such saturation is to calculate the requisite thickness of wall, as if the earth were a fluid, making $\phi = 0$ in the formulæ.

Another way of providing against such a contingency is to construct, sloping against the back of the wall, a bank of shivers of stone or of coarse gravel, whose angle of repose is not affected by the presence of water, and then to fill in the softer material. The pressure against the wall in this case will not at any time greatly exceed that of a bank of the firm material employed, sloping at its own angle of repose.

The subject of relieving retaining walls from pressure by the aid of arches, and that of securing their foundations by special contrivances in swampy ground, will be considered further on.

The cope of a retaining wall should consist of large flat stones laid as headers.

272. **Land Ties for Retaining Walls.**—Retaining walls having to bear a great pressure, while they rest on a yielding foundation, may have their stability increased by being tied or anchored by iron rods to vertical or nearly vertical plates of iron imbedded in a firm stratum of earth at a distance behind the wall sufficient

to prevent its being disturbed by the operations of excavation, building, and embanking, connected with the erection of the wall.

The holding power, per foot of breadth, of a rectangular vertical anchoring plate, the depths of whose upper and lower edges below the surface are respectively x_1 and x_2, may be approximately calculated from the following formula:—

Let w' be the weight of a cubic foot of the earth;

ϕ' its angle of repose;

H, the holding power per foot of breadth; then

$$H = w' \cdot \frac{x_2^2 - x_1^2}{2} \cdot \frac{4 \sin \phi}{\cos^2 \phi} \text{ ....(1.)}$$

The depth of the centre of pressure of the plate below the surface of the ground is given by the following expression:—

$$\frac{2}{3} \cdot \frac{x_2^3 - x_1^3}{x_2^2 - x_1^2}; \text{ ....(2.)}$$

and to that centre the tie-rod should be attached.

If the retaining wall depends on the tie-rods alone for its security against sliding forward, they should be fastened to plates on the face of the wall in the line of the resultant pressure of the earth behind the wall, that is, at one-third of the height of the wall above its base. But if the resistance to sliding forward is to be distributed between the foundation and the tie-rods, they are to be placed at a higher level; for example, if half the horizontal thrust is to be borne by the foundation, and half by the tie-rods, the latter should be fixed to the wall at two-thirds of its height above the base.

273. **Struts for Retaining Walls.**—The base of a retaining wall may be prevented from sliding forward by a series of horizontal struts of masonry or brickwork, abutting against rectangular masses whose resistance to displacement depends on the same principles with the holding power of anchoring plates, stated in the last article.

When a cutting in soft ground has a retaining wall at each side of it, the foundations of the walls may be kept asunder, and thus prevented from sliding forward, by means of a series of inverted arches extending between them, across and below the base of the cutting, so as to act as transverse struts.

The upper parts of such walls may also be held asunder by slightly arched ribs of cast iron or of brick. These ribs abut against the walls at about two-thirds of their height above their

274. Relieving Arches.—A row of arches having their axes and the faces of their piers at right angles to the face of a bank of earth are called "relieving arches." There may be either one or several tiers of them, and their front ends may be closed by a vertical wall, which thus presents the appearance of a retaining wall, although the length of the archways is such as to prevent the earth from abutting against it. Fig. 175 represents a vertical transverse section of such a wall, with two tiers of relieving arches behind it. To compute the length of a relieving arch from its clear height, or its clear height from its length, the following approximate formulæ may be used, in which

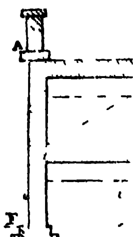


Fig. 175

x denotes the depth of the crown of an arch below the surface,
h, its clear height,
l, its length, and
ϕ, the angle of repose of the earth.

$$l = \operatorname{cotan} \phi \left(h + \frac{x}{(1 + \sin \phi)^2}\right); \quad \ldots \ldots \quad (1.)$$

$$h = l \cdot \tan \phi - \frac{x}{(1 + \sin \phi)^2} \quad \ldots \ldots \ldots \ldots \quad (2.)$$

To determine the conditions of stability of such a structure as a whole, the horizontal pressure against the vertical plane O D may be determined, and compounded with the weight of the combined mass of masonry and earth O A E D in front of that plane, to find the resultant pressure on the base.

In soft ground the bases of the piers of the lowest tier of relieving arches should be connected by means of inverted arches, so as to distribute the pressure over the whole area covered by the structure.

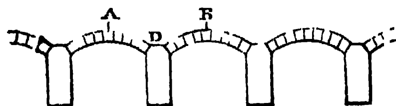


Fig. 176.

275. Buttressed Horizontal Arches.—Fig. 176 represents a plan, or horizontal section, of part of a row of buttresses, connected by horizontally arched walls.

To find the thickness, T = D E, required for such buttresses, let

B denote A B, the breadth of the mass of earth which one buttress has to sustain;
b, the breadth of the buttress:

t, the thickness which would be required for an uniform wall, to sustain the same bank of earth, computed as in Article 266, equation 2, p. 405; then

$$T \quad t \cdot \sqrt{\;} \qquad b = B \cdot \frac{t^2}{T^2}. \qquad (1.)$$

In soft ground the bases of the buttresses may be connected by means of inverted arches, to distribute the pressure; and their tops may, if necessary, be connected by means of arches, in order to support a platform, or a surcharged bank of earth In the last-mentioned case, t is to be computed as in Article 270, p. 409.

Section VIII.—*Of Stone and Brick Arches.*

276. **General Structure of Arches of Stone.** (*A. M.*, 223.)—An arch of masonry consists of a sector of a ring, composed of courses of wedge-formed stones, called *arch-stones* or *voussoirs*, pressing against each other at surfaces called *bed-joints*, which are, or ought to be, perpendicular, or nearly perpendicular, to the *soffit*, or internal concave surface of the arch. The soffit is also called, in mathematical language, the *intrados*. The word *extrados* is applied sometimes to the upper surface of the ring of arch-stones; sometimes to that of the solid masonry or *backing* above them; sometimes to that of the entire mass of permanent loading material. (See also p. 203.) The outer or convex surface of the ring of arch-stones, which may be either a curved surface or a series of steps, sustains the vertical pressure of that part of the load which arises from the weight of materials other than the arch-stones themselves; and that outer surface also exerts in many cases a horizontal or inclined thrust against the *spandrils* and *abutments*. The abutments sustain also the thrust of the lowest voussoirs, vertical or inclined, as the case may be. The course of stones from which an arch springs is called the *springing-course* or *skew-back*, the latter term being used when its upper and lower beds are oblique to each other. Sometimes an arch springs at once from the ground, so that its abutments are its foundations.

A wall standing on an arch, in the plane of the arch-ring, is called a *spandril wall*. The arch of a bridge requires a pair of *external spandril walls*, one over each *face* of the arch; the space between them is filled up to a certain level with solid masonry, and above that level it is sometimes filled with earth or rubbish, and sometimes occupied by a series of *internal spandril walls* parallel to the external spandril walls, and having vacant spaces between them—a mode of construction favourable both to stability and to

lightness. In order to form a continuous platform for the roadway, the spaces between the internal spandril walls are sometimes covered with flags of some strong stone (such as slate), and sometimes arched over with small transverse arches. The external spandril walls are the abutments of those arches, and must have stability sufficient to sustain their thrust: when the spandrils are filled with earth or rubbish, the external spandril walls must have stability sufficient to withstand the pressure of the filling material.

277. **Masonry of Arches—Backing.**—The description of masonry used for arches is either ashlar or block-in-course; the beds being perpendicular, or nearly perpendicular to the direction of the thrust through the arch-ring, and the side-joints perpendicular to the beds and to the soffit. In common, or *square* arches, in which the axis of the archway is perpendicular to each face of the arch, the bed-joints are plane; in oblique, or *skew* arches, they are curved surfaces, shaped according to principles which will be explained in a later article.

The principles according to which the masonry of arches is to be built are in other respects the same with those already explained in Articles 212 and 213, pp. 384 to 386. Special care is to be taken to cut and lay the beds of the stones accurately, and to make the bed-joints thin and close, in order that the arch may be strained as little as possible by settling. To insure this, some engineers have caused arches to be built dry, *grout* or liquid mortar being afterwards run into the joints; but the advantage of this method is doubtful. Others have placed sheets of lead in the bed-joints, to distribute the pressure between the stones.

The *backing* of an arch consists of block-in-course, coursed rubble, or random rubble, and sometimes of concrete. When the backs of the arch-stones are cut into steps, the backing is built in courses of the same depth with those steps, and thus bonded with them. Sometimes the backing is built in radiating courses, whose beds are prolongations of the bed-joints of the arch-stones. Both these methods are favourable to strength and stability.

The height to which the solid backing should be built is regulated by principles which will be explained in subsequent articles.

The upper surface of the backing, and of that part of the arch, if any, near the crown, which is without backing, is *coated* with a layer of waterproof material, such as clay puddle (Article 206, p. 344) mixed cement (Article 231, p. 374), or bituminous concrete (Article 234, p. 376); the last being the best. Any rain-water which penetrates the structure above the arch flows to the valleys or lowest parts of this coating, whence it is carried away by tubes or other convenient outlets.

278. **Brick Arches** may be built either of gauged or wedge-formed bricks, moulded or rubbed so as to suit to the radius of the soffit, or of bricks of the common shape. In the former case, the methods of bonding the bricks are the same with those employed in walls (Article 255, p. 394); in the latter, the bricks are accommodated to the curved figure of the arch by making the bed-joints thinner towards the intrados than towards the extrados, or, if the curvature is sharp, by driving thin pieces of slate into the outer edges of those joints; and different methods are followed for bonding them. The most common way is to build the arch in concentric rings, each half-a-brick thick: this is, in fact, to lay the bricks all stretchers, and to depend upon the tenacity of the mortar or cement for the connection of the several rings. It is deficient in strength, unless the bricks are laid in cement at least as tenacious as themselves. Another way is to introduce courses of headers at intervals, so as to connect pairs of half-brick rings together. This may be done either by thickening the joints of the outer of a pair of half-brick rings with pieces of slate, so that there shall be the same number of courses of stretchers in each ring between two courses of headers; or by placing the courses of headers at such distances apart that between each pair of them there shall be one course of stretchers more in the outer than in the inner ring. The former method is the best suited to arches of long radius, the latter to those of short radius.

Hoop-iron bond (Article 256, p. 395), laid *round the arch*, between half-brick rings, as well as longitudinally and radially, is very useful for strengthening brick arches. The bands of hoop-iron which traverse the arch radially may be bent, and prolonged in the bed-joints of the backing and spandrils. By the aid of hoop-iron bond Sir Marc-Isambard Brunel built a half-arch of bricks laid in strong cement, which stood projecting from its abutment like a bracket, to the distance of 60 feet, until it was destroyed by its foundation being undermined.

279. **Use of Centres.**—A centre is a temporary structure of timber or iron (but most commonly of timber), by which the voussoirs of an arch are supported until the arch is completed, and capable of supporting itself. The principles of the strength, stability, and construction of centres will be explained under the head of CARPENTRY.

A centre consists of parallel frames or *ribs* about 5 or 6 feet apart, curved on the outside to a figure parallel to that of the soffit of the arch, and supporting a series of transverse planks called *laggings*, upon which the archstones directly rest.

The oldest and most common kind of centre is one which can be lowered or "struck" all in one piece, by driving out wedges from

below it, so as to remove the support from below every part of the arch at once. An improved kind was first introduced by Hartley, in which *each lagging* is supported upon the ribs by wedges or screws of its own, so that the support can be removed from the arch-stones course by course, and replaced in the event of the settlement proving too rapid.

The centre of an arch should not be struck until the solid part of the backing has been built, and the mortar has had time to set; and when an arch forms one of a series of arches, with piers between them, no centre should be struck so as to leave a pier with an arch abutting against one side of it only, unless the pier is what is termed an "abutment-pier;" that is, a pier which has sufficient stability to act as an abutment.

280. **Line of Pressures in an Arch—Condition of Stability.** (*A. M.*, 224.)—If a straight line be drawn through each bed-joint of the arch-ring, representing the position and direction of the resultant of the pressure at that joint, the straight lines so drawn form a polygon, and each of the angles of that polygon is situated in the line of action of the resultant external force acting on the arch-stone which lies between the pair of joints to which the contiguous sides of the polygon correspond; so that the polygon is similar to a polygonal frame, loaded at its angles with the forces which act on the arch-stones (their own weight included). A curve inscribed in that polygon, so as to touch all its sides, is the *line of pressures* of the arch. The smaller and the more numerous the arch-stones into which the arch-ring is subdivided, the more nearly does the polygon coincide with the curve; and the curve or line of pressures represents an ideal *linear arch*, which would be balanced under the continuously-distributed forces which act on the real arch under consideration. From the near approach of this linear arch to the polygon whose sides traverse the centres of resistance of the bed-joints, the points where the linear arch cuts those joints may be taken without sensible error for the centres of resistance.

Now in order that the stability of the arch may be secure, it is necessary that no joint should tend to open either at its outer or at its inner edge; and in order that this may be the case, the centre of resistance of each joint should not deviate from the centre of the joint by more than one-sixth of the depth of the joint; that is to say, the centre of resistance should lie within the *middle third* of the depth of the joint; whence follows this

THEOREM. *The stability of an arch is secure, if a linear arch, balanced under the forces which act on the real arch, can be drawn within the middle third of the depth of the arch-ring.* It is through this theorem that the principles of the stability of ideal linear arches or ribs, already explained in Article 132, p. 202, and the

previous articles referred to in that article, and also in Articles 133 to 139, pp. 203 to 218, become applicable to real arches of masonry and brickwork. (See Addenda, p. 790.)

It may be held that in most practical examples the tenacity of fresh mortar is not sufficiently great to be taken into account in determining the stability of masonry; and hence, where cement is not used, all horizontal or oblique conjugate forces which maintain the equilibrium of the arch-ring must be pressures, acting on the arch from without inwards The linear arch, therefore, is limited in such cases to those forms which are balanced under *pressures from without alone;* that is to say, that the intensity of the horizontal or conjugate pressure, denoted by p_y in Article 138, equation 4, p. 214, *must not at any point be negative*

It is true that arches have stood, and still stand, in which the centres of resistance of joints fall beyond the middle third of the depth of the arch-ring; but the stability of such arches is either now precarious, or must have been precarious while the mortar was fresh.

When tenacity to resist horizontal or oblique tension is given to the spandrils of an arch, and to the joints between them and the arch-stones, by means of cement, hoop-iron bond, iron cramps, or otherwise, the conjugate tension denoted by $-p_y$ must not at any point exceed a safe proportion of that tenacity; that is to say, about one-eighth By this means stability may be given to arches of seemingly anomalous figures; but such structures are safe on a small scale only.

281. **Relation between Linear Rib and Intrados of Real Arch.**—There are numerous cases in which the form of a linear rib, suited to sustain a given load, may at once be adopted for the intrados of a real arch for sustaining the same load, with sufficient exactness for practical purposes. The following is the test whether this method is applicable in any given case. Let A C B in fig. 177 be one half of the ideal rib which it is proposed to adopt as the intrados of a real arch. Draw A a normal to the rib at the crown, so as to represent a length not exceeding two-thirds of the intended depth of the keystone, and conceive a couple applied to the keystone, consisting of tension at A equal and opposite to the thrust along the rib there, and of an equal thrust at a. Draw a normal B b at the springing, and make

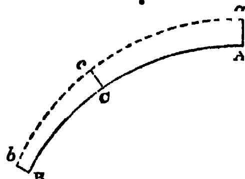


Fig. 177.

$$\frac{\text{B}\,b}{\text{A}\,a} = \frac{\text{thrust along rib at A}}{\text{thrust along rib at B}};$$

and conceive a couple of equal moment to the first, consisting of

tension at B and thrust at b, to be applied at the springing. The pair of couples thus introduced, being equal and opposite, do not alter the equilibrium, their only effect being to transfer the line of pressures from the intrados or ideal rib A C B to a line $a\ c\ b$, whose perpendicular distance C c from the intrados at any point is inversely as the thrust along the rib at that point. Then if $a\ c\ b$ lies within the middle third of the arch-ring, the ideal rib A C B is of a suitable form for the intrados.

The process may, if required, be commenced by making B b not less than one-third of the depth of the arch at the springing, that depth having first been fixed with due regard to the conditions of resistance to crushing, which will be considered further on.

282. **Use of Equilibrated or Transformed Catenary Arches.**—The transformed catenary has already been fully described in Article 131, pp. 200 to 202. When used for the intrados of an arch, it is commonly called the "curve of equilibrium." It is suited for the support of any load whose pressure is wholly vertical, and whose extrados is either a horizontal plane coinciding with the *directrix* O X of the transformed catenary (fig. 113), or another transformed catenary having the same directrix.

When the method of Article 281 is applied to an intrados of this figure, the resulting curve, $a\ c\ b$ of fig. 177, is simply the same curve shifted vertically upwards through the height A a.

In making use of the transformed catenary in practice, there are usually given, the directrix, the crown of the arch, and the points from which it is to spring. From these data the *modulus* m is to be computed by means of equation 4 of Article 131, p. 202; and then the vertical ordinate y from the directrix at any given horizontal distance x from the crown of the arch is given by the formula,—

$$y = \frac{y_0}{2}\left(e^{\frac{x}{m}} + e^{-\frac{x}{m}}\right); \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (1.)$$

in which y_0 is the depth of the curve of the arch below the directrix. (See table, Article 298, p. 436.)

In applying the formulæ (3) of Article 131, p. 202, to this arch, *thrust* is to be read instead of *tension;* and the symbol w is to be understood to stand for *the load per square foot of the vertical area between the intrados and the directrix.*

For example, let B denote the breadth of the arch, or length of the archway, in feet; let w_1 be the weight of a cubic foot of the masonry; let the extrados be a transformed catenary, each of whose ordinates, measured from the directrix, is equal to the corresponding ordinate of the intrados multiplied by a fraction n; and let a fraction k be the ratio of the volume of solid building to the

whole volume, the remainder $1 - k$ consisting of spandril-voids. Then,

$$w = n\,k\,w_1. \quad \text{.........................(2.)}$$

In order that the arch may be equilibrated under its own weight and that of the solid backing alone, as well as when the whole structure is finished, the figure of the upper surface of the solid backing should itself either be a transformed catenary, or approximate to that curve.

283. **Use of the Hydrostatic Arch.**—The mathematical and mechanical properties of this arch, considered as a linear rib, have been explained in Article 136, pp. 208 to 212.

Inasmuch as the thrust through this arch is uniform, the application of the method of Article 281 to it produces simply a curve parallel to it; so that if it be used for the intrados of an arch-ring of uniform thickness, and the centre of resistance at the keystone be at the middle of the thickness, the line of pressures will be at the middle of the thickness of the arch-ring throughout, or approximately so. The word "approximately" is used, because the thrust along the real arch is not exactly uniform, like that in the ideal rib; for at the springing it is greater than at the crown, by an amount equal to the weight of the prism of masonry which stands vertically above the springing-course; but that difference is practically unimportant.

The application of the hydrostatic arch to practice is founded on the fact, that every arch, after having been built, subsides at the crown, and spreads, or tends to spread, at the haunches, which therefore press horizontally against the filling of the spandrils; from which it is inferred as probable, that if an arch be built of a figure suited to equilibrium under fluid pressure—that is, pressure of equal intensity in all directions—it will spread horizontally, and compress the masonry of the spandrils, until the horizontal pressure at each point becomes of equal intensity to the vertical pressure, and therefore sufficient to keep the arch in equilibrio.

In addition to the methods already explained in Article 136 for drawing the figure of the arch, the following method may be given for describing an approximation to it about five centres. It is very simple, and has been found by trial to answer well.

In fig. 178, let F B be the half-span and F A the rise of the proposed arch. Make A C $= \rho_0$, and B D $= \rho_1$, the radii of curvature at the crown and springing, as calculated by the formulæ (11 and 12) of Article 136, p. 211.* Then C will be one of the

* The formulæ for computing those radii may be put in the following form: let a be the rise; y_1 the half-span—

$$b = y_1 + \frac{y_1^2}{80a}; \quad \rho_0 = \frac{a}{2}\left(1 + \frac{b^4}{a^4}\right); \quad \rho_1 = \frac{a}{2}\left(1 + \frac{a^2}{b^2}\right).$$

centres, and D another. About D, with the radius D E = F A — B D, describe a circular arc, and about C with the radius C E = C F, describe another circular arc; let E be the point of intersection of those arcs; this will be a third centre; and two more centres will be similarly situated to D and E with respect to the other half-arch.

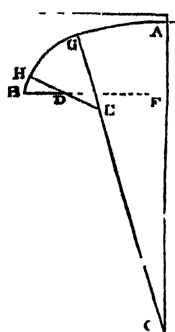


Fig. 178.

Then about C with the radius C A, draw the circular arc A G till it cuts C E produced in G; about E, with the radius E G = F A, draw the circular arc G H till it cuts E D produced in H; about D, with the radius D B, draw the circular arc H B. This completes one half-arch, and the other is drawn in the same manner.

The curve thus described falls a little beyond the true hydrostatic arch at G, and a little within it at H.

To the greatest possible security to a hydrostatic arch, especially if the span is great compared with the rise, the backing ought to be built of solid rubble masonry up to the level of the crown of the extrados, before the centre is struck.

Many *semi-elliptic arches* may be treated as approximate hydrostatic arches. In fact, many of the arches called semi-elliptic approximate more nearly to the figure of the hydrostatic arch than to that of the true semi-ellipse. The true semi-ellipse of a given span and rise differs from the hydrostatic arch by being of somewhat sharper curvature at the crown and springing, and somewhat flatter curvature at the haunches, and by enclosing a somewhat less area.

284. **Use of the Geostatic Arch.**—The derivation of the figure of this arch, by transformation from that of the hydrostatic arch, and most of its properties, have been explained in Article 137, pp. 212, 213. The following problem only remains to be solved. Given in a geostatic arch, the rise a, the half-span s, and the depth of load at the crown x_0; it is required to find the proportion c, which the half-span and other horizontal dimensions bear to the corresponding dimensions of a hydrostatic arch whose vertical dimensions are the same:—

Make

$$b = a \cdot \left(\frac{x_0 + a}{x_0}\right)^{\frac{1}{3}}; \text{ and } y_1 = \frac{b^2}{30\,a}; \text{ then}$$

$$c = \frac{s}{y_1} \text{ nearly.} \qquad (1.)$$

This method may be applied to those *semi-elliptic arches* which are not approximate hydrostatic arches.

285. Stability of any proposed Arch. (*A. M.*, 225.)—The first step towards determining whether a proposed arch will be stable, is to *assume* a linear arch parallel to the intrados or soffit of the proposed arch, and loaded vertically with the same weight, distributed in the same manner. The *size* of this assumed linear arch is a matter of indifference, provided each point in it is considered as subjected to the same forces which act at the *corresponding joint* in the real arch; that is, *the joint at which the inclination of the real arch to the horizon is the same with that of the assumed arch at the given point.*

The assumed linear arch is next to be treated according to the method of Article 138, pp. 213 to 216; and by equation 4 of that Article, p. 214, is to be determined, either a general expression, or a series of values, of the intensity p_y of the conjugate pressure, horizontal or oblique, as the case may be, required to keep the arch in equilibrio under the given vertical load. If that pressure is nowhere negative, a curve similar to the assumed arch, drawn through the middle of the arch-ring, will be either exactly or very nearly the line of pressures of the proposed arch; p_y will represent, either exactly or very nearly, the intensity of the lateral pressure which the real arch, tending to spread outwards under its load, will exert at each point against its spandril and abutments; and the thrust along the linear arch at each point will be the thrust of the real arch at the corresponding joint.

On the other hand, if p_y has some negative values for the assumed linear arch, there must be a pair of points in that arch where that quantity changes from positive to negative, and is equal to nothing. The angle of inclination i_0 at that point, called the *angle of rupture*, is to be determined by solving Problem IV. of Article 138, pp. 215, 216. The corresponding joints in the real arch are called the *joints of rupture;* and it is below those joints that conjugate pressure from without is required to sustain the arch, and that consequently the backing must be built with squared side joints.

In fig. 179, let B C A represent one-half of a symmetrical arch, K L D E an abutment, and C the joint of rupture, found by the method already described. The *point* of rupture, which is the centre of resistance of the joint of rupture, is somewhere within the middle third of the depth of that joint; and from that point down to the springing joint B, the line of pressures is a curve similar to the assumed linear arch, and

Fig. 179.

parallel to the intrados, being kept in equilibrio by the lateral pressure between the arch and its spandril and abutment.

From the joint of rupture C to the crown A, the figure of the true line of pressures is determined by the condition, that it shall be a linear arch balanced under vertical forces only; that is to say, the horizontal component of the thrust along it at each point is a constant quantity, and equal to the horizontal component of the thrust along the arch at the joint of rupture.

That horizontal thrust, denoted by H_0, is found in solving Problem IV., Article 138, p. 215, and is the horizontal thrust of the entire arch.

[If the arch is distorted, *conjugate thrust* is to be read instead of "*horizontal thrust*," wherever that phrase occurs.]

The only point in the line of pressures above the joints of rupture which it is important to determine, is that which is at the crown of the arch, A, and it is found in the following manner:—

Find the centre of gravity of the load between the joint of rupture C and the crown A; and draw through that centre of gravity a vertical line.

Then if it be possible, from one point, such as M, in that vertical line, to draw a pair of lines, one parallel to a tangent to the soffit at the joint of rupture, and the other parallel to a tangent to the soffit at the crown, so that the former of those lines shall cut the joint of rupture, and the latter the keystone, in a pair of points which are both within the middle third of the depth of the arch-ring, the stability of the arch will be secure; and if the first point be the point of rupture, the second will be the centre of resistance at the crown of the arch, and the crown of the true line of pressures.

When the pair of points related to each other as above do not fall at opposite limits of the middle third of the arch-ring, their exact positions are to a small extent uncertain; but that uncertainty is of no consequence in practice. Their most probable positions are equi-distant from the middle line of the arch-ring.

Should the pair of points fall beyond the middle third of the arch-ring, the depth of the arch-stones must be increased.

286. **Circular Arch, not less than a Quadrant.**—In applying the principles of the preceding article to an arch whose intrados is an arc of a circle, it is necessary to modify the equations 9 and 11 of Article 138, Example IV., p. 217, in order to take into account the weight of a given portion of the ring of arch-stones, as distinguished from that of the material which rests upon it. The results are as follows:—

In a circular arch with a horizontal platform above it, let

r denote the radius of the intrados;

r' that of the extrados of the *arch-ring*, which is supposed uniformly thick;

c the depth of loading material above the crown of the arch;

w the weight of a cubic foot of the arch-stones;

w' the *mean weight* of a cubic foot of the superstructure, including voids ($= \frac{1}{2} w$ nearly);

Then the solution of the following equation gives the *angle of rupture* i_0;

$$0 = \left\{ \frac{w'}{w} (1 - \cos i_0) + \left(1 - \frac{w'}{w}\right) \frac{i_0 - \cos i_0 \sin i_0}{2 \sin^3 i_0} \right\} r'^2$$

$$+ \frac{w'}{w} c\, r' - \frac{i_0 - \cos i_0 \sin i_0}{2 \sin^3 i_0} r^2; \quad \ldots\ldots\ldots\ldots\ldots (1.)$$

which having been solved, the following equation gives the *horizontal thrust for each unit of breadth of the arch:*—

$$H_0 = w' r'^2 \left\{ \left(1 + \frac{c}{r'}\right) \cos i_0 - \frac{\cos^2 i_0}{2} - \frac{i_0 \operatorname{cotan} i_0}{2} \right\}$$

$$+ w (r'^2 - r^2) \frac{i_0 \operatorname{cotan} i_0}{2}. \quad \ldots\ldots\ldots\ldots\ldots\ldots (2.)$$

Equation 1 is here given in the form best suited for practical use, being that of a quadratic equation between r', r, and c, with co-efficients depending on the angle i_0, and on the comparative heaviness of the arch and of the superstructure. The value of i_0 can be calculated from a given set of values of r', r, c, and $\frac{w'}{w}$, by a series of trials only; but if a value of i_0 be assumed, then any one of those four quantities can be computed exactly by solving the quadratic equation.

PROBLEM.—*The radius of the intrados* r, *and the height* c *of the horizontal platform above the crown of the arch being given, to find the outer radius* r' *of the arch-ring corresponding to an assumed angle of rupture* i_0. In this case r' is the unknown quantity; and if equation 1 be denoted for brevity's sake by

$$A r'^2 + \frac{w'}{w} c\, r' - B r^2 = 0,$$

its solution is

$$r' = \sqrt{\left(\frac{B}{A} r^2 + \frac{w'^2 c^2}{4 w^2 A^2}\right)} - \frac{w'}{2 w A}. \quad \ldots\ldots\ldots (3.)$$

In most of the examples of circular arches which occur in practice, the angle of rupture lies between 45° and 55°; so that if the squared backing is carried up to that part of the arch which is inclined at an angle of 45° to the horizon, its height will be sufficient, at least.

It further appears, by trial, that the following approximate rule seldom errs by so much as one-twentieth part in giving the horizontal thrust of the arch.

The horizontal thrust is nearly equal to the weight supported between the crown and that part of the soffit whose inclination is 45°; or in symbols,

$$H_0 \text{ nearly} = w' r' (\cdot 0644 r' + \cdot 7071 c) + \cdot 3927 w (r'^2 - r^2) \ldots (4.)$$

Thus, in fig. 180, let A C B represent one-half of a circular arch, O being the centre of the intrados and O A its radius, $= r$; let O P $= r'$, P U $= c$; U V being the horizontal platform. Draw O C F, making the angle A O C = 45° with the vertical; then the horizontal thrust of the arch will be nearly equal to the weight of the mass A C F V U, which lies between the joint C F and the crown. The point F is that up to whose level it is advisable to build the backing solid, or, at all events, to bond and joint it in such a manner that it shall be capable of transmitting a horizontal thrust. Draw F T horizontal; then

Fig. 180

P T = ·7071 O P

287 **Stability of an Unloaded Arch-Ring.**—Before the centre of an arch is struck, the spandril walls (or spandril filling, as the case may be) should be built to such a height that the part of the arch-ring near the crown, which is left for the time unloaded, shall fulfil the condition of stability which consists in having the line of pressures within the middle third of its thickness, when under its own weight alone. The exact investigation of this question is very complex, and it is unnecessary to give it here in detail. An approximation sufficiently accurate for practical purposes is as follows:—

Make the depth of the lowest point of the extrados of the unloaded arc below its highest point a mean proportional between the thickness of the arch and the radius of the intrados.

That is to say, in fig. 180 (last Article) make

$$P Q = \sqrt{\overline{P A} \cdot \overline{A O}} = \sqrt{r' r - r^2}; \qquad (1.)$$

then the horizontal line Q R S will show the level up to which the spandril walls or spandril filling are to be built before the centre is struck.

288. **Circular Arch less than a Quadrant.**—In this case the rule of the preceding article is to be applied exactly as in the case of an arch not less than a quadrant; but in computing the horizontal thrust, it is sufficient to take the weight of a half-arch with its load, and multiply by the co-tangent of the inclination of the intrados to the horizon at the springing.

289. **Tie-Walls**, in the hollow spandrils of arches, are transverse walls at right angles to the spandril walls. The distance from centre to centre of the tie-walls may be from three to five times the distance from centre to centre of the spandril walls.

290. **Depth of Keystone.**—To determine with precision the depth required for the keystone of an arch by direct deduction from the principles of stability and strength, would be an almost impracticable problem from its complexity. That depth is always many times greater than the depth necessary to resist the direct crushing action of the thrust. The proportion in which it is so in some of the best existing examples has been calculated, and found to range from 3 to 70. The smaller of these factors may be held to err on the side of boldness, and the latter on the side of caution; good medium values are those ranging from 20 to 40. The best course in practice is to assume a depth for the keystone according to an empirical rule, founded on dimensions of good existing examples of bridges.

The following is such a rule:—

For the depth of the keystone, take a mean proportional between the radius of curvature of the intrados at the crown, and a constant, whose values are,

for a single arch,........... .... ·12 foot;
for an arch forming one of a series,.. . . . . ·17 „

That is to say, in symbols,
Depth of keystone for a single arch,

$$\text{in feet} = \sqrt{(\cdot 12 \times \text{radius at crown})}. \quad \text{............(1.)}$$

Depth of keystone for an arch of a series,

$$\text{in feet} = \sqrt{(\cdot 17 \times \text{radius at crown})}. \quad \text{............(2.)}$$

The following are examples:—

SINGLE ARCHES.	RADIUS AT CROWN. Feet.	DEPTH OF KEYSTONE. Calculated Feet.	Actual Feet.
Bridge over the Severn, at Gloucester (by Telford); elliptic arch; span 150 feet, rise 35 feet, ...	160·7	4·39	4·5
Bridge at Turin, over the Dora Riparia (by Mosca); arch segmental; span 147·6 feet, rise 18 feet, ...	160	4·38	4·9
Grosvenor Bridge, over the Dee, at Chester (by Hartley and Harrison); segmental arch, span 200 feet; rise 42 feet, .	140	4·1	4·0
Ordinary bridge over a double line of railway; elliptic arch; span 30 feet; rise 7 feet 6 inches,.. ...	30	1·9	1·83 to 2
ARCHES IN SERIES.			
Bridge over the Thames, near Maidenhead (by I. K. Brunel); arch (of brick in cement) nearly elliptic, span 128 feet, rise 24·25, ...	169	5·36	5·25
London Bridge (by Sir John Rennie); elliptic arch; span 152 feet, ...	162	5·25	5·00
Bridge of Neuilly (by Perronet), basket-handle arch, span 39 mètres = 128 feet nearly, rise 9·75 mètres = 32 feet nearly,	159	5·20	5·13
Bridge of St Maxence (by Perronet); segmental arch; span about 76·7 feet; rise about 6·4 feet, ...	119	4·49	4·79
Waterloo Bridge (by Rennie); elliptic arch; span 120 feet; rise 32 feet,............................	112·5	4·37	5·00
Ballochmyle Bridge, over the Ayr, (by Miller); semicircular arch; span 181 feet; rise 90·5 feet, ...	90·5	3·92	4·5
Dean Bridge, near Edinburgh; segmental arch; span 90 feet; rise 30 feet,............................	48·75	2·88	3·00

It is evident from the law approximately followed by the examples in the preceding table, that the depth required for the arch-ring is regulated chiefly by the necessity for providing against deviations of the line of pressures, produced by temporary partial loads; and because such loads on a large arch are less as compared with the weight of the arch itself than in a small arch, the depth of the arch-stones increases more slowly than the general dimensions of the arch—viz., proportionally to the square root of the radius at the crown.

The *probability* of such a rule being found to answer in practice might have been inferred from equation 38 A of Article 180, p. 307, by assuming that, owing to the plasticity of the mortar, the dead load of the arch, there denoted by w_0, produces no bending action (which is equivalent to omitting the term in B); and then determining the depth h of the arched rib so that the tension p'_1 shall be $= 0$, the arch being considered as sensibly flexible *between the joints of rupture only*. This last condition makes the rise k of the sensibly flexible part of the arch equal to a certain fraction of the radius at the crown; say $n\,r$, so that the equation referred to is reduced to the following:—

$$0{\cdot}138\,\frac{w}{w_0\,q'\,h} - \frac{1}{n\,r} = 0.$$

But $q' = \frac{1}{6}$; and $w. \div w_0$, which expresses the ratio of the intensity of the external load to the weight of the arch itself, may be replaced by $h' \div h$; h being the required depth of keystone, and h' the depth of the same material which is equivalent to the external load. The equation thus becomes,

$$0{\cdot}828\,\frac{h'}{h^2} - \frac{1}{n\,r} = 0;\ \text{or}$$

$$h^2 = {\cdot}828\,n\,h'\,r; \qquad (3.)$$

that is to say, *for equally intense external loads, and equal angles of rupture, the square of the thickness of the keystone should vary as the radius of the intrados;* being very nearly the rule deduced empirically from practical examples. The co-efficients ·12 and ·17 in equations 1 and 2 correspond to the factor ·828 $n\,h'$ in equation 3. It is probable that the necessity for a larger co-efficient in the case of an arch which forms one of a series arises from the fact, that when one arch of a series is loaded externally, and the adjoining arches unloaded, the piers yield slightly, so as to lower the position of the joints of rupture.

291. **An Abutment in Radiating Courses** forms in truth a continuation of the arch, and is the strongest and most stable kind of

abutment where the foundation is firm, and the height from which the arch springs is moderate. One of the best examples is the Grosvenor Bridge at Chester. The real face of such an abutment is a continuation of the intrados of the arch, and its back is a continuation of the extrados of the solid spandril backing, which ought to be built in radiating courses also; but the custom is, in violation of good taste, to disguise the real structure by means of a casing of masonry with vertical and horizontal joints.

292. **Vertical Abutments** depend for their stability on the same principles which regulate that of buttresses, and which have been fully explained in Articles 263 and 264, pp. 396 to 401. The points to be chiefly attended to are, that if there is any horizontal thrust through the spandril, the part of the abutment above the springing of the arch shall have sufficient weight to resist by friction the tendency to sliding produced by that thrust; that above the bed-joint next below the springing of the arch, the weight of material, including that of the half-arch itself with its load, shall produce friction enough to resist the whole thrust of the arch, whether exerted through the spandrils or at the springing; and that the centre of resistance at the base of the abutment shall not deviate from the centre of the base by more than the proper fraction, q, of the thickness of that base. (See Article 263, p. 396, and Article 179, pp 294, 295.)

It is highly advantageous, in point both of stability and economy, to build abutments with hollows in them, or with narrow archways passing through them, perpendicular to the main archways which the abutments support. These archways should have inverted arches at the bottom, to distribute the load over as large a base as possible. The hollows or archways may occupy about one-third of the whole volume of the abutment.

When an arch, as in fig. 179, p. 421, has a joint of rupture such as C, the part of the arch below that joint, together with its spandril backing and the load directly resting on it, may be considered as forming part of the abutment.

In some of the best examples of bridges, the thickness of the abutments ranges from *one-third* to *one-fifth* of the radius of curvature of the arch at its crown.

293. **Piers of Arches.**—Each pier of a series of arches ought to have sufficient stability to resist the thrust which acts upon it when one only of the arches which spring from it is loaded with a travelling load. That thrust may be roughly computed by multiplying the travelling load per lineal foot by the radius of curvature of the intrados at its crown in feet.

Each pier should always give sufficient space on its top for the two arches to spring from.

Either of these rules gives in general a less thickness than those adopted for piers in practice, which range from *one-tenth* to *one-fourth* of the span of the arches; the latter thickness, and those approaching to it, being suitable for "*abutment-piers.*" The most common thickness, for ordinary piers, is from one-sixth to one-seventh of the span of the arches.

Piers, like abutments, are advantageously lightened, especially when very lofty, as in viaducts, by being built hollow, or by having archways traversing them, with inverted arches at the base.

294. **Ribbed Arches, Abutments, and Piers.**—Arches and their abutments and piers may be made at once light and stiff, by building them in parallel deep ribs, with thinner portions of masonry between them; but this of course involves additional workmanship.

295. **Skew Arches** are of figures derived from those of symmetrical arches by distortion in a horizontal plane. The elevation of the face of a skew arch, and every vertical section parallel to its face, being similar to the corresponding elevation and vertical section of a symmetrical arch, the forces which act in a vertical layer or rib of a skew arch with its abutments, are the same with those which act in an equally thick vertical layer of a symmetrical arch with its abutments, of the same dimensions and figure, and similarly and equally loaded.

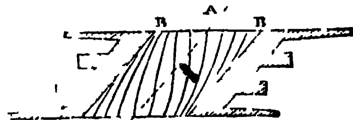


Fig. 181.

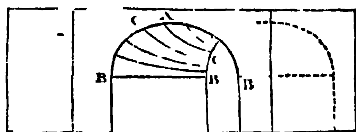


Fig. 182.

Fig. 181 represents a plan of a skew arch, with counterforted abutments. The *angle of skew*, or *obliquity*, is the angle which the axis of the archway, A A, makes with a perpendicular to the face of the arch, B C A B. The span of the archway, "*on the square*," as it is called (that is, the perpendicular distance between the abutments), is less than the span *on the skew*, or parallel to the face of the arch, in the ratio of the cosine of the obliquity to unity. It is the span *on the skew* which is equal to that of the corresponding symmetrical arch.

The best position for the bed-joints of the arch-stones is perpendicular to the thrust along the arch. If, therefore, there be drawn on the soffit of a skew arch, a series of parallel curves, made by the intersections of the soffit with vertical planes parallel to the face of the arch, the best forms for the bed-joints will be a series of

curves drawn on the soffit of the arch so as to cut the whole of the former series of curves at right angles, such as C C in figs. 181 and 182. Joints of the best form being difficult to execute, spiral joints are used in practice as an approximation.

Preparatory to the execution of a skew arch, a large drawing of the soffit must be prepared, showing the exact figure and position of every arch-stone. That drawing represents the curved surface of the soffit as if it were *spread out flat*, and is called the "*development*" of that curved surface. In general it is sufficient to draw one-half of the soffit, the other half being similar. The following are the processes by which that drawing is prepared:—

1. *To draw the development of the soffit, and of its vertical sections on the skew.* Fig. 183, No. 2, represents a plan of one half of the arch, H A K being the crown of the soffit, and I B L the face of

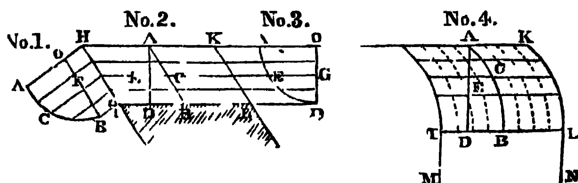


Fig. 183.

one of the abutments. The line A C B represents the position of a vertical section *on the skew*, and A E D, perpendicular to H K, that of a vertical section *on the square*: ∠ B A D being the angle of obliquity.

Assume any convenient number of points in H I, through which draw a set of lines (such as E C E G) ∥ H K, and also a set of lines ⊥ H I. Draw O B ∥ H I, cutting these lines; and on O B as half-span, construct the vertical section of the arch *on the skew*, represented by No 1, in which A C B is the line on the soffit corresponding to A C B in No. 2.

Construct the vertical section *on the square*, No. 3, by drawing O D ∥ A D to represent the half-span on the square, and transferring the ordinates of No. 1 to the corresponding points in No. 3; for example, F C to G E.

Then construct the development No. 4 in the following manner:—Produce the centre line of the soffit, H A K A O H A K. From any convenient point A, No. 4, draw A E D ⊥ H K, in which take distances A E, A D, &c., equal in length to the arcs A E, A D, &c., which are cut off on the curve A E D, No. 3, by its several ordinates. Then will the straight line A E D, No. 4, be the development of the *section on the square*, A E D, Nos. 2 and

3. Through the points of division of A E D, No. 4, draw lines parallel to H K, such as E C, I D B L, &c., on which lay off ordinates, such as E C, D B, &c., equal respectively to the corresponding ordinates, E C, D B, &c., in the plan, No. 2, and through the ends of those ordinates draw a curve A C B, No. 4; this will be the development of the vertical section *on the skew*, A C B, Nos. 1 and 2.

Draw also the curves H I, K L parallel, similar, and equal to A C B, and at distances from it on either side, H A = A K, of half the length of the archway. Then I H K L will be the development of half the soffit. Draw I M and L N perpendicular to I L; then M I L N will be the development of part of the face of an abutment. Draw also any convenient number of intermediate curves, such as those shown by dots, parallel, similar, and equal to A C B, to represent the development of several parallel skew vertical sections of the soffit, and to indicate, at the same time, the direction of the thrust at each point which they traverse. These may be called "*curves of pressure*."

II. *To draw on the development of the soffit, the bed-joints and side-joints of true courses.* The bed-joints are drawn by sketching with the free hand a series of curves, cutting all the curves of pressure at right angles, and called the *orthogonal trajectories of the curves of pressure.* The side-joints, being perpendicular to the bed-joints, are parts of curves of pressure themselves. (See fig. 184.) The courses become thinner towards the acute angle of the abutment, and thicker towards the obtuse angle, so that it may be sometimes advisable to introduce intermediate bed-joints near the obtuse angle, as shown near L in fig. 184. In the illustrations, the arch springs vertically from the abutments, so that none of the bed-joints intersect the line of springing, I L, to which they are all asymptotes. Had the arch been segmental, some of the bed-joints would have intersected that line obliquely, making necessary the use of skew-backs of the kind shown in the next figure, but not so oblique.

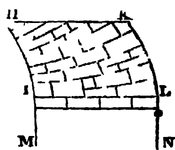


Fig. 184.

III. *To draw, on the development of the soffit, the bed-joints and side-joints of spiral courses.* (See fig. 185.) On the development of the soffit, draw a series of parallel equidistant straight lines, perpendicular to the direction of the thrust at the crown of the arch; these will represent the bed-joints, and the side-joints will be perpendicular to them. Between I and L are shown the *skew-backs*, or stones which connect the slanting courses of the arch with the horizontal courses of the abutment.

Spiral courses are perpendicular to the thrust at the crown of

the arch only, and become more and more oblique to it the nearer they are to the springing.

296. **Ribbed Skew Arch.**—A substitute for an ordinary skew arch is sometimes composed of a series of ribs placed side by side,

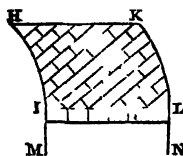
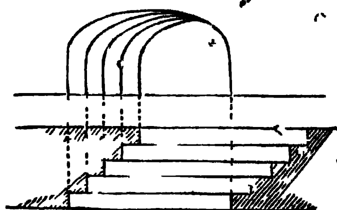


Fig. 185


Fig. 186.

as in fig. 186. This mode of construction contracts the span *on the square* as compared with that of an ordinary skew arch having the same span *on the skew*, by the following quantity:—

Let a denote the projection of each rib of the abutment beyond the preceding rib,

b, the breadth of a rib; then

$$\text{contraction of span on the square} = \frac{a\,b}{\sqrt{a^2 + b^2}} \ldots\ldots (1.)$$

If the span *on the square* has already been fixed, the *span on the skew* for a ribbed arch is to be made greater than that for a common skew arch, simply by the projection a of a rib of the abutment

297. **Strength of Stone and Brick Arches.**—A well-designed stone or brick arch of sufficient stability has usually a great surplus of strength. In the case, however, of a proposed arch of unusual dimensions or figure, especially if the material is comparatively weak, it is advisable, after the figure and dimensions have been planned with a view to stability, to test whether the strength is likely to be sufficient This may be done with a sufficiently close approximation, by the aid of equations 30, 36, and 37 of Article 180, p. 307, by making the following substitutions:—

CASE I. *In a transformed Catenary Arch, or a circular segment of less than a quadrant;* make

h' = depth of arch-stones at springing × cos inclination of arch at that point;

$$q = \frac{1}{6};\ m' = \frac{1}{2};\ l = \text{span},\ k = \text{rise};$$

$$\text{and consequently, } \mathrm{B} = \frac{15}{16}\frac{h'^2}{k^2}\left(1 + \frac{16}{3}\frac{k^2}{l^2}\right); \ldots\ldots\ldots (1.)$$

$$\therefore \quad r_1 = \frac{2}{5} \cdot (1 + B) \left(1 - \frac{h'}{4k}\right); \ldots\ldots\ldots\ldots\ldots (2.)$$

also, considering a rib of a foot in breadth, make $A_1 = h'$; for w_0 put the dead load in lbs. per square foot of platform *at the crown of the arch*, and for w the rolling load in lbs. per square foot of platform.

CASE II. *In hydrostatic, elliptic, and semicircular arches, and circular segments greater than a quadrant*, make $l =$ the span, and $k =$ the rise, of the part of the arch lying between the two joints which make angles of 45° with the horizon. In hydrostatic and elliptic arches, make $h' =$ thickness of arch-ring; in circular arches, make $h' =$ thickness at the joints first mentioned $\times$ ·707. Then proceed in other respects as in Case I.

In all cases omit the term $\frac{l^2 w_0}{8 A_1 (1 + B) k}$, and substitute for it $H \div h'$, H being the horizontal thrust as found by the methods of Articles 132 to 138, pp. 202 to 218, Article 286, pp. 423, 424, and Article 288, p. 425, for a rib one foot in breadth.

Equation 37, giving the greatest intensity of thrust, p_1, in *lbs. on the square foot*, thus becomes

$$p_1 = \frac{H}{h'} + \frac{l^2}{2 h'^2} \left\{ \frac{w_0 B}{1 + B} + w \left(2 r_1^2 - 2 r_1^3 + \frac{3}{5} r_1^4\right) \right\}. (3.)$$

When $h' \div k$, and consequently B, are small fractions, so that $r_1 = \frac{2}{5}$ nearly, the following formula may be used:—

$$p_1 = \frac{H}{h'} + \frac{l^2}{2 h'^2} \left\{ B w_0 + 0.207 w \right\}. \ldots\ldots (3 \text{ A.})$$

When, on the other hand, $(1 + B) \div \left(1 - \frac{h'}{4 k}\right)$ is equal to or greater than $\frac{5}{2}$, so that $r_1 = 1$, the following formula is to be used:—

$$p_1 = \frac{H}{h'} + \frac{l^2}{2 h'^2} \left\{ \frac{B w_0}{1 + B} + 0.288 w \right\}. \ldots\ldots (3 \text{ B.})$$

In transformed catenarian and circular segmental arches, an approximation to uniformity of strength may be obtained by making the depth of each voussoir proportional to the secant of the inclination of its bed to the vertical.

297 A. **Underground Archways—Tunnels—Culverts.**—If the depth of a buried archway, such as a tunnel or culvert, beneath the surface of the ground, is great compared with the height of the archway, the proper form for the line of pressures, which must lie within the

middle third of the thickness of the arch, is the elliptic linear arch of Article 134, p. 202, in which the ratio of the horizontal to the vertical semi-axis is the square root of the ratio of the horizontal to the vertical pressure of the earth; that is to say,

$$\frac{\text{horizontal semi-axis}}{\text{vertical semi-axis}} = c = \sqrt{\frac{p_u}{p_v}} = \sqrt{\left(\frac{1 - \sin \phi}{1 + \sin \phi}\right)}; \text{(1.)}$$

ϕ being the angle of repose.

If the earth is firm, and little liable to be disturbed, the proportion of the half-span, or horizontal semi-axis, to the rise, or vertical semi-axis, may be made *greater* than is given by the preceding equation, and the earth will still resist the additional horizontal thrust; but that proportion should never be made *less* than the value given by the equation, or the sides of the archway will be in danger of being forced inwards.

In a drainage tunnel or culvert the entire ellipse may be used as the figure of the arch; but in a railway tunnel, where it is necessary to have a flat floor, the sides and roof of the tunnel comprise in height the upper two thirds, or three-fourths, of the ellipse, which is closed below by a circular segmental inverted arch of slight curvature, its depression being one eighth of its span, or thereabouts. By this mode of construction the vertical pressure of the sides of the tunnel is concentrated upon foundation courses directly below them, from which they spring. The ratio which the entire width of the tunnel, measured *outside* the masonry or brickwork, bears to the joint width of that pair of foundations, must not exceed the limit of the ratio of the weight of a building to the weight of earth displaced by it, as given by Article 237, equation 1, p. 379. The inverted arch serves to prevent the foundations of the sides of the tunnel from being forced inwards by the horizontal pressure of the earth.

The *exact* form for the line of pressures in the sides and roof of a tunnel is the *geostatic arch* of Article 137, pp. 212, 213. This principle requires attention when the roof of the tunnel is near the surface. Let x_0 be the depth of the crown of the tunnel, and x_1 that of its greatest horizontal diameter, beneath the surface. From those ordinates as data, design a *hydrostatic arch*, by the aid of the formulæ (12) of Article 136, p. 211; contract the horizontal ordinates of that arch in the ratio $c : 1$ (see equation 1, above); and the result will be the figure of the geostatic arch required.

The greatest intensity of pressure in a buried archway occurs usually in its sides, at the ends of the shorter diameter of the oval intrados; and that intensity is given approximately by the following equation. Let x_1 be the depth of the shorter diameter below the surface of the ground, b' the half-span of the archway, a' its

rise, t the thickness of its side, w the weight of a cubic foot of the earth; then the greatest pressure, *in lbs. on the square foot*, is

$$q = \frac{w\{x_1(b'+t) - 0\cdot 8\, a'b'\}}{t}; \quad\ldots\ldots\ldots\ldots(2.)$$

and this should not exceed the resistance of the material to crushing, divided by a proper factor of safety.

It appears that in the brickwork of various existing tunnels, the factor of safety is as low as *four*. This is sufficient, because of the steadiness of the load; but in buried archways exposed to shocks like those of culverts under high embankments, the factor of safety should be greater; say from *eight* to *ten*.

How small soever the load may be, there is a certain minimum thickness for an underground archway, for determining which the following empirical rule, exactly similar to that for finding the depth of the keystone of an arch, has been deduced from practical examples. The rise and half-span being denoted as before by a' and b', compute approximately the longest radius of curvature of the intrados by the formula

$$b' \quad\quad .(3.)$$

then

$$\text{least thickness } t \text{ in feet } \sqrt{0\cdot 12\, r}. \ldots\ldots\ldots\ldots(4.)$$

This is applicable where the ground is of the firmest and safest kind. In soft and slippery materials, the thickness ranges from *once and a-half* to *double* that given by equation 4; that is to say,

$$\text{from } \sqrt{0\cdot 27\, r} \text{ to } \sqrt{0\, 48\, r}.\ldots\ldots\ldots\ldots(4\ \text{A.})$$

The thickness of an underground arch at the crown may be made less than at the sides in the ratio $b' : a'$; but the more common practice is to make it uniform.

As to the precautions to be observed in embanking over and near archways, see Article 201, p. 341.

298. **Table of Co-ordinates and Slopes of Catenarian Curves.**—(See Article 131, pp. 200, 201, and Article 282, p. 418.)

Let m denote the modulus, or parameter;

y_0, the ordinate from the directrix to the crown;

x, any abscissa, measured horizontally from the crown;

y, the corresponding ordinate from the directrix;

$\frac{dy}{dx}$, the tangent of the slope of the curve at the end of that ordinate.

$\frac{x}{m}$	$\frac{y}{y_0}$	$\frac{m}{y_0}\frac{dy}{dx}$	$\frac{x}{m}$	$\frac{y}{y_0}$	$\frac{m}{y_0}\frac{dy}{dx}$
0	1·0000	·0000	1·6	2·5774	2·3755
0·2	1·0200	·2013	1·8	3·1074	2·9421
0·4	1·0810	·4107	2·0	3·7622	3·6269
0·6	1·1854	·6366	2·2	4·5679	4·4571
0·8	1·3373	·8880	2·4	5·5569	5·4662
1·0	1·5431	1·1752	2·6	6·7690	6·6947
1·2	1·8106	1·5094	2·8	8·2526	8·1918
1·4	2·1509	1·9043	3·0	10·0676	10·0178

To interpolate the ordinate $y \pm v$ corresponding to an intermediate abscissa $x \pm u$, when $\frac{y}{y_0}$ corresponds to $\frac{x}{m}$ in the table, make

$$\frac{y \pm v}{y_0} = \frac{y}{y_0}\left(1 + \frac{u^2}{2\,m^2} + \frac{u^4}{24\,m^4}\right) \pm \frac{m}{y_0}\frac{d\,y}{d\,x}\left(\frac{u}{m} + \frac{u^3}{6\,m^3}\right). \quad (1$$

This computation is to be performed by addition to the number next below in the table, or by subtraction from the number next above, according as the intermediate abscissa lies nearer to the one next below it or to that next above it.

When very great precision is not required, the terms in u^3 and u^4 may be neglected; but those in u and u^2 should *always* be computed. The greatest possible error within the limits of the table by using equation 1 as it stands, is about ·00005; by neglecting u^3 and u^4, that limit of error is increased, for the greatest intermediate ordinate in the table, to about ·0015.

298 A. **List of Authorities on Masonry.**—(Stones, Limes, and Cements)—Berthier, *Traité des Essais par la Voie sèche,* Vicat, *Traité des Mortiers,* Pasley on *Cements and Mortars.* (Masonry in general)—Rondelet, *Traité de l'Art de Bâtir,* Gauthey, *Traité de la Construction des Ponts,* Tredgold on *Masonry* (*Encyc. Brit.*); Reid's *Portland Cement, and Practical Treatise on Concrete Making;* Faija, *Portland Cement;* Gilmore, *Limes, Hydraulic Cements, and Mortars;* Redgrave, *Calcareous Cements* (1895).

Addendum to Article 230, p. 374.

Iron Concrete (introduced by Mr. Leslie) is composed of 17 parts by weight of gravel, and 1 part of iron turnings spread in alternate layers. It is used in sea-works. The iron becomes oxidated by degrees, and in the course of two or three months cements the gravel into a hard mass.